INVESTMENT AND ARBITRAGE OPPORTUNITIES WITH SHORT SALES CONSTRAINTS

LAURENCE CARASSUS

CEREMADE (Université de Paris Dauphine) and CREST

ELYÈS JOUINI

CREST-ENSAE and CERMSEM (Université de Paris 1 Panthéone Sorbonne)

In this paper we consider a family of investment projects defined by their deterministic cash flows. We assume stationarity—that is, projects available today are the same as those available in the past. In this framework, we prove that the absence of arbitrage opportunities is equivalent to the existence of a discount rate such that the net present value of all projects is nonpositive if the projects cannot be sold short and is equal to zero otherwise. Our result allows for an infinite number of projects and for continuous as well as discrete cash flows, generalizing similar results established by Cantor and Lippman (1983, 1995) and Adler and Gale (1997) in a discrete time framework and for a finite number of projects.

KEY WORDS: investment, short sales constraint, stationarity, arbitrage, Radon measure, Laplace transform

1. INTRODUCTION

In this paper, we consider a model in which agents face investments opportunities (or investments) described by their cash flows as in Gale (1965), Cantor and Lippman (1983, 1995), Adler and Gale (1997), and Dermody and Rockafellar (1991, 1995). These cash flows can be at each time positive as well as negative. It is easy to show that such a model is a generalization of the classical one with financial assets. As in Cantor and Lippman and Adler and Gale, we will show that the absence of arbitrage opportunities is equivalent to the existence of a discount rate such that the net present value of all projects is nonpositive. We will extend this result in two directions: (i) allowing our model to contain an infinite number of investments and (ii) allowing the cash flows to be continuous as well as discrete, which is never the case for all the mentioned references.

The model we consider assumes absence of risk, stationarity, and short sales constraints. In the general theory of arbitrage formalized by Harrison and Kreps (1979), Harrison and Pliska (1981), and Kreps (1981), securities markets are assumed to be frictionless, and the main result is that the absence of arbitrage opportunities (or no arbitrage) is equivalent to the existence of an equivalent martingale measure. The existence of state prices follows. In our framework, we prove that there exist some state prices with the particular form: e^{-rt}.

Address correspondence to Carassus at CREST, 15 Boulevard Gabriel Peri, 92245 Malakoff Cedex, France; or to Jouini at CREST-ENSAE, 15 Boulevard Gabriel Peri, 92245 Malakoff Cedex, France.

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In our model we allow short sales constraints, but only in order to give an intuition of our result. Let us consider a simple frictionless setup. The absence of arbitrage opportunities implies the existence at any time t of a positive discount function D_t, where $D_t(s)$ is the market value at time t of one dollar paid at time $t+s$ (in discrete settings this is just an implication of the separating hyperplane theorem). No arbitrage means no arbitrage even for contracts that may not be present, including forward contracts and zero coupon bonds. Following Cox, Ingersoll, and Ross (1981), the consequence of the no arbitrage condition in a deterministic setting is that the spot bond price is equal to the forward bond price. So the forward price at time t of a bond delivered at time $t+s$ and paying one dollar at time $t+T$ for $s<T$ (that is, $D_t(T)/D_t(s)$) is equal to the price at time $t+s$ of one dollar paid at time $t+T$, $D_{t+s}(T-s)$. Roughly speaking, stationarity in the model would imply stationarity for D; i.e. $D_t = D_{t+s}$ for all t and s. Hence we get $D(T)/D(s) = D(T-s)$, and the solution to this equation is $D(t) = e^{-rt}$ for some constant r. In fact, the stationarity for D is not straightforward and we prove that there exist a set of discount functions containing an exponential.

In this paper, we assume that every investment is available in every period of the investment horizon. This means that we can begin the investment at each date—this is called stationarity. We will also assume that an investor can only hold a positive number of investments in each time period. This is an economic constraint. Otherwise, according to Cantor and Lippman (1983), it would be possible "to build a negative number of bridges or harvest a negative number of forests." So we will impose here that no investment can be sold. Note that this short-selling constraint could be a restriction for some financial assets. But, in fact, our model also includes the case without constraints (see Corollary 3.1). In our model, the time horizon is not finite. The investor goal is to become rich in a finite time, but this one is not specified at the beginning. So we will ask the investments and the strategies to end in a finite time.

2. THE MODEL

In the discrete case, an investment project m is characterized by $(m_0, \ldots, m_T)$ where the real number m_t represents the cash received from the project in the tth period. A negative m_t means that the investor must pay for the project. Similarly, for a positive m_t, the investor receives payment from the project. In this formalization, it does not matter if assets have a price or not. If m_0 is negative it could represent the price to pay in order to ensure later the cash flows $m_1, \ldots, m_T$. Here, we choose to include the price in the cash flow sequence. So, we will consider that this sequence has price zero: an investor subscribes or not to the sequence at no cost.

In this paper, an investment is represented as a Radon measure (for example, see Bourbaki (1965) or Rudin (1966)).

Recall that if we denote by E_n the space of continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$ and with support included in $[-n, n]$ and if we attribute to E_n the topology $\mathcal{T}_n$ of the uniform convergence on $[-n, n]$ (E_n is a classical Banach space) then we can attribute to $E = \cup_{n\in\mathbb{N}} E_n$ the strict inductive limit topology $\mathcal{T}$ (because in this case the topology induced by $\mathcal{T}_{n+1}$ on E_n is the same as $\mathcal{T}_n$). This strict inductive limit topology $\mathcal{T}$ is defined such that for all n the topology induced by $\mathcal{T}$ on E_n is the same as $\mathcal{T}_n$. The completeness of E is shown in Bourbaki (1987), and we recall that with this topology on E, the space

E^* of continuous linear forms on E is the Radon measure space. The weak-* topology on the space E^* of the Radon measures is called the vague topology, and a sequence (π_n) of Radon measures converges vaguely to π if for all function $\varphi \in E$, $\pi_n(\varphi)$ converges to $\pi(\varphi)$. Notice that, using one of the Riesz representation theorems, a positive Radon measure is uniquely associated with a Borel–Radon measure, and we will use the same notation for both of them.

Roughly, for an investment represented by a Radon measure μ, $\int_{[t_1,t_2)} d\mu$ represents the investment payment between times t_1 and t_2. The description via Radon measures allows us to describe investments with discrete as well as continuous cash flows in a unified way. Under this terminology, the preceding discrete payment $m = (m_0, \ldots, m_T)$ is represented by the discrete measure $\mu = \sum_{t=0}^{T} m_t \delta_t$, where δ_t is the Dirac measure in t. But it also allows us to treat investments having continuous payoff—that is, investments represented by a function m. In this case, $m(t)dt$ should be interpreted as the investment payment in the short period dt. The Radon measure μ associated with this investment is given by the following measure defined by a density $d\mu(t) = m(t)dt$.

We allow our model to contain an infinite number of investments. For example, to model an interest rate in a continuous setup, we need an infinite number of investments: one should consider all the possible repayment dates. The set of investment income streams is modeled by a family of Radon measure $(\mu_i)_{i \in I}$ with I infinite. Each investment i has a finite horizon T_i and we assume that the support of measure μ_i lies in $[0, T_i]$. This assumption is necessary because if we assume the existence of an investment with an infinite horizon, it will always be possible to suspend repayment of the debt to infinity. This is not an arbitrage opportunity because the investor wants to become rich in a finite time. In this model, the investor is only allowed to choose a finite number of investments. Among an infinite number of possibilities, there are only a finite number of choices. To make the model clear, let us consider the case of a single discrete investment m. At each time t, we must choose the number of subscriptions λ_t to investment m. At time 0, we buy λ_0 investments, which assures a payoff of $\lambda_0 m_0$. At time 1 the total payoff is $\lambda_0 m_1 + \lambda_1 m_0$, and at time $t \leq T$ it will be $\lambda_0 m_t + \lambda_1 m_{t-1} + \cdots + \lambda_{t-1} m_1 + \lambda_t m_0$, which can be described by the convolution product $\lambda * m(t)$. In the general case, after selecting a finite subset J of the set I of investments, the investor chooses the number of subscriptions from each element of J. For the same reasons as before, these numbers will be modeled by a family $(\lambda_j)_{j \in J}$ of Radon measures. Roughly, $\int_{[t_1,t_2)} d\lambda_i$ represents the number of investments i bought between times t_1 and t_2. We also require that the support of all measures λ_i is in a fixed compact set. Moreover, the no-sell assumption requires all the λ_i to be positive. The previous payoff calculus is easily generalized and the choice of a finite subset J of I and a strategy $(\lambda_j)_{j \in J}$ leads to the payoff $\sum_{j \in J} \lambda_j * \mu_j$.

The following example, from Adler and Gale (1997), shows that it is possible to make an arbitrarily large profit in a finite time. Consider an investment that pays \$1 today. The investor must pay \$2 tomorrow and finally receives \$1.01 the day after. We denote this investment by $m = (1, -2, 1.01)$. As previously, the investor has no money to begin with, so the only way to pay the second day's installment on a unit of investment is by initiating a second investment at level two. It is straightforward to show that in order to get a zero payoff, the investor must subscribe at time t to $\lambda_t = -(\lambda_{t-2} m_2 + \lambda_{t-1} m_1)$ investments. A simple calculus leads to a positive payoff after 32 periods. So with this investment it is possible to become arbitrarily rich after 32 periods (assuming one can buy an arbitrarily large number of investments m).

3. CHARACTERIZATION OF THE EXISTENCE OF AN ARBITRAGE OPPORTUNITY

3.1. Definitions and Main Results

As we saw before, a strategy is defined as follows:

DEFINITION 3.1. A strategy is defined by the choice of:

- a finite subset J of I,
- an investment horizon $n \geq \max_{j \in J} T_j$,
- a buying strategy for the set of investments J modeled by a family of nonnegative Radon measure $(\lambda_j)_{j \in J}$ such that the support of λ_j is included in $[0, n - T_j]$, for all $j \in J$.

Next, we define the absence of arbitrage opportunities.

DEFINITION 3.2. There is an arbitrage if and only if there exists a strategy $(\lambda_j)_{j \in J}$ such that the corresponding payoff $\sum_{j \in J} \lambda_j * \mu_j$ is a nonnegative and nonzero measure π.

We want to show that the absence of arbitrage opportunities is equivalent to the existence of a discount rate, such that the net present value of all projects is nonpositive. To prove this, we will assume that there exists at least one investment that is positive at the beginning, and another at the end. Note that if we consider a discrete time model or even a continuous time model, this condition seems to be quite natural. If all the investments are negative at the beginning, it is straightforward to see that the payoff associated with a nonnegative strategy is necessarily negative at the beginning and then there is no arbitrage possibility. The same can be applied at the end and our condition seems therefore to be redundant. In fact, some particular situations are excluded by such a reasoning: the case of investments with oscillations in the neighborhood of the initial or final date such that we cannot define a sign to the investment at these dates. Nevertheless, our condition is justified if we admit that such situations are pathological.

We say that a measure μ_k (resp. μ_ℓ) is positive in zero (resp. T_ℓ) if there exists a positive real ε_k (resp. ε_ℓ) such that for all continuous and nonnegative function φ with support contained in $[-\varepsilon_k, \varepsilon_k]$ (resp. $[T_\ell - \varepsilon_\ell, T_\ell + \varepsilon_\ell]$), positive in zero (resp. in T_ℓ), the integral $\int \varphi \, d\mu_k$ (resp. $\int \varphi \, d\mu_\ell$) is positive.

ASSUMPTION 3.1. *There exist at least two investments k and ℓ with $T_l \leq T_k$, such that the measure μ_k is positive in zero and the measure μ_ℓ is positive in T_ℓ.*

In the following, if Assumption 3.1 holds, we will call ε the infimum of ε_k and ε_ℓ. We will see that if the model contains a borrowing and a lending rate, Assumption 3.1 is always satisfied (see Corollary 3.2). We will also see that Assumption 3.1 is useless in the case of a discrete setup.

Under Assumption 3.1, our main result is stated as follows.

THEOREM 3.1. *Under Assumption 3.1, the absence of arbitrage opportunities is equivalent to the existence of a discount rate r such that for all i in I, the net present value $\int e^{-rt} \, d\mu_i(t)$ is nonpositive.*

In the sequel, we will see that we need only Assumption 3.1 to show the first implication.

We will see in step 5 that Theorem 3.1 means that there is an arbitrage opportunity if and only if there exists a finite subset J of I, such that $\sup_{j\in J}\int e^{-rt}\,d\mu_j(t)$ is positive for all rates r. Furthermore, if we add for all investments μ_i the investment $-\mu_i$ in the model, we obtain the situation where all investments can be sold, and the proof of the following result becomes straightforward.

COROLLARY 3.1. *If all investments can be either bought or sold, under Assumption 3.1 the absence of arbitrage opportunities is equivalent to the existence of a discount rate r, such that for all i in I, the net present value $\int e^{-rt}\,d\mu_i(t)$ is equal to zero.*

We recall that the lending rate r_0 (resp. the borrowing rate r_1) is the rate at which the investor is allowed to save (resp. to borrow). A lending rate is modeled by the following family of investments $\mu_t = -\delta_0 + e^{r_0 t}\delta_t$ (you lend one dollar at time zero and you will get back $e^{r_0 t}$ at all the possible repayment dates t). Similarly, a borrowing rate can be represented as the family ${\mu'}_t = \delta_0 - e^{r_1 t}\delta_t$. Note that for all $t > 0$, ${\mu'}_t$ is positive in 0 and μ_t is positive in t, and both have support included in $[0, t]$. Assumption 3.1 is satisfied with the investments μ_{T_ℓ} and ${\mu'}_{T_k}$, where T_ℓ and T_k are some reals with $T_k \geq T_\ell$.

COROLLARY 3.2. *If there exists a lending rate r_0 and a borrowing rate r_1, the absence of arbitrage opportunities is equivalent to the existence of a discount rate r included in $[r_0, r_1]$ such that for all i in I, the net present value $\int e^{-rt}\,d\mu_i(t)$ is nonpositive.*

Proof of the Theorem. We begin the proof showing that the absence of arbitrage opportunities implies the existence of a discount rate r, such that for all i in I, the net present value $\int e^{-rt}\,d\mu_i(t)$ is nonpositive. So we will assume that the family $(\mu_i)_{i\in I}$ does not constitute an arbitrage opportunity.

Step 1: *From measures to continuous functions.*
In order to work in a space that displays good properties, we want to use functions instead of measures, so we are going to "transform" our financial world in order to have only functions. In fact, we will work in the space E of continuous functions with compact support. For this purpose, we will use a function g mapping $\mathbb{R}^+$ into $\mathbb{R}$ with support equal to $[0, \varepsilon]$, positive on $[0, \varepsilon)$ and continuous on $\mathbb{R}^+$. We will denote by $(f_i)_{i\in I}$, the family of convolution product $f_i = \mu_i * g$. It is straightforward to show that the support of f_i is now $[0, T_i + \varepsilon]$, and we will note $\tilde{T}_i = T_i + \varepsilon$, the horizon of the "transform" investment. Let J be a finite subset of I and n be an integer; we denote by F_n^J the set defined by $F_n^J = \{\sum_{j\in J}\lambda_j * f_j/\lambda_j \geq 0$ with support in $[0, n - \tilde{T}_j]\}$ and $P = \{g \in E/g \geq 0\}$. It is clear then that the absence of arbitrage opportunities implies that $P \cap F_n^J = \{0\}$.

It is straightforward to show that the function f_k is positive on $]0, \varepsilon[$, and that the function f_ℓ is positive on $]T_\ell, \tilde{T}_\ell[$. The proof of this result is left to the reader.

Step 2: *Use of a separating hyperplane theorem.*
In order to separate P and F_n^J, we will consider the set $\tilde{E}_n$ of continuous functions mapping $[0, n]$ into $\mathbb{R}$ and endowed with the uniform convergence topology. Note that $\tilde{E}_n$ is different from the set of restrictions to $\mathbb{R}^+$ of functions of E_n because E_n is a set of continuous functions with support in $[-n, n]$. This implies that $f(n) = 0$ for all $f \in E_n$ restricted to $\mathbb{R}^+$, which is not the case in $\tilde{E}_n$.

Let us define $P_n = \{g \in \tilde{E}_n : g \geq 0\}$. It is easy to see that F_n^J can be seen as a subset of $\tilde{E}_n$ and that F_n^J and P_n are convex sets. Furthermore, we know that $P \cap F_n^J = \{0\}$. It is then sufficient to remark that P_n has a nonempty interior, which does not contain 0 (these points are left to the reader) in order to apply the separation theorem given by Dunford and Schwartz (1958, p. 417). This theorem is relative to two disjoint convex sets (here, F_n^J and the interior of P_n), one of which has a nonempty interior. By continuity of the separation functional, the separation result will hold for F_n^J and P_n.

We obtain then a nonzero measure ν_n^J on $[0, n]$ which can be identified to a measure on $\mathbb{R}$ equal to the initial one on $[0, n]$ and equal to zero elsewhere. We will also call ν_n^J this new measure. It is easy to see that $\nu_n^J \in E^*$ and separates, in fact, P and F_n^J. More precisely, we have

$$\forall (f, g) \in F_n^J \times P, \qquad \nu_n^J(f) \leq 0 \leq \nu_n^J(g).$$

Step 3: *Normalization of the measures* ν_n^J.
Let τ be defined by $\tau = \sup\{u / \int_{[0,u)} d\nu_n^J = 0\}$. Because ν_n^J is nonnegative and nonzero with support in $[0, n]$ it follows that $\tau \leq n$.

Assume that $n \geq \tilde{T}_\ell$ and that τ is in $[\tilde{T}_\ell, n[$. It is straightforward to show that there exists a u in $[\varepsilon, n - \tilde{T}_\ell]$ such that τ is in $[T_\ell + u, \tilde{T}_\ell + u[$. Let $(\lambda_j)_{j\in J}$ be the strategy defined by $\lambda_\ell = \delta_u$ (the Dirac measure at u), and $\lambda_j = 0$ for $j \in J$ and $j \neq l$. The payoff associated with this strategy is clearly in $[0, n]$ and consequently $\nu_n^J(\delta_u * f_\ell)$ is nonpositive. Noting that $\nu_n^J(\delta_u * f_\ell) = \int f_\ell(x-u)\, d\nu_n^J(x)$ and using the definition of τ and of the support of f_ℓ, we obtain that $\nu_n^J(\delta_u * f_\ell) = \int_{[\tau, \tilde{T}_\ell + u]} f_\ell(x-u)\, d\nu_n^J(x)$. If x is in $[\tau, \tilde{T}_\ell + u]$, then $x - u$ is in $[\tau - u, \tilde{T}_\ell]$, which is included by definition of u in $[T_\ell, \tilde{T}_\ell]$. We know that f_ℓ is positive on the interior of this last interval. So the nonpositivity of the considered integral implies that the support of ν_n^J does not intersect $[\tau, \tilde{T}_\ell + u]$, which contradicts the definition of τ. We can conclude then that $\tau \leq \tilde{T}_\ell$. Let b be a nonnegative continuous function equal to 1 on $[0, T_\ell + \varepsilon/2]$ and with support equal to $[0, \tilde{T}_\ell]$. It is clear that we can impose $\nu_n^J(b) = 1$ or, equivalently, $\int b(x)\, d\nu_n^J(x) = 1$.

We will prove now that there exists a constant ρ, depending only on the investments, such that for all finite subset J of I and for all $t \in [0, n-\varepsilon]$, $\int_{[0,t)} d\nu_n^J \leq \rho^t$. As previously done, applying the separation result to the strategy using only the investment f_ℓ, in quantity $\lambda_\ell = \delta_u$, for u in $[0, n - \tilde{T}_\ell]$, we obtain $\int f_\ell(x-u)\, d\nu_n^J(x) \leq 0$. We know that f_ℓ is positive on $]T_\ell, \tilde{T}_\ell[$ and consequently there exists a positive real number A such that $f_\ell(t) \geq A$ for all $t \in [T_\ell + \varepsilon/3, T_\ell + 2\varepsilon/3]$. The quantity $\int f_\ell(x-u)\, d\nu_n^J(x) = \int_u^{u+\tilde{T}_\ell} f_\ell(x-u)\, d\nu_n^J(x)$ is nonpositive, using the definition of A, and the fact that f_ℓ is nonnegative on $[T_\ell + 2\varepsilon/3, T_\ell + \varepsilon]$. It follows that

$$\begin{aligned} A \int_{u+T_\ell+\varepsilon/3}^{u+T_\ell+2\varepsilon/3} d\nu_n^J \leq \int_{u+T_\ell+\varepsilon/3}^{u+T_\ell+\varepsilon} f_\ell(x-u)\, d\nu_n^J(x) &\leq -\int_u^{u+T_\ell+\varepsilon/3} f_\ell(x-u)\, d\nu_n^J(x) \\ &\leq \|f_\ell\|_\infty \int_u^{u+T_\ell+\varepsilon/3} d\nu_n^J. \end{aligned}$$

Thus, we get

$$\int_{u+T_\ell+\varepsilon/3}^{u+T_\ell+2\varepsilon/3} d\nu_n^J \leq \frac{\|f_\ell\|_\infty}{A} \int_u^{u+T_\ell+\varepsilon/3} d\nu_n^J.$$

Applying the previous inequality to $u = (N-1)\varepsilon/3$, we find that

$$\int_0^{T_\ell+(N+1)\varepsilon/3} d\nu_n^J \le \left(1 + \frac{\| f_\ell \|_\infty}{A}\right) \int_0^{T_\ell+N\varepsilon/3} d\nu_n^J.$$

Ranging from $N = 1$ to $(N-1)\varepsilon/3 \le n - \tilde{T}_\ell$, we obtain for all t in $[0, n-\varepsilon]$, $\int_0^t d\nu_n^J(x) \le \rho^t$, with $\rho = (1 + (\| f_\ell \|_\infty)/A)^{3/\varepsilon}$.

We will denote by $\nu_n^{'J}$ the following Radon measure, $\nu_n^{'J}(\varphi) = \int \varphi I_{[0,n-\varepsilon]}\, d\nu_n^J$, for all functions φ in E. If φ is a continuous function with support in $[a, b]$ then it is easy to see that $\nu_n^{'J}(\varphi) \le \rho^b \|\varphi\|_\infty$. Following Bourbaki (1965), a sufficient condition is to show that for all finite subsets J of I, $\{\nu_n^{'J}/n \in \mathbb{N}\}$ is a vaguely relatively compact set. Consequently, we can assume that the sequence $(\nu_n^{'J})$ converges vaguely to some measure ν^J, and it follows immediately that the sequence (ν_n^J) converges to the same limit. From the definition of the weak-* topology and from the fact that $\nu_n^J(b) = 1$, we obtain that $\nu^J(b) = 1$, and therefore ν^J is nonzero. Following the same approach, it is straightforward to show that ν^J is nonnegative.

Finally, for all nonnegative Radon measures λ with compact support included in $\mathbb{R}^+$, $\int f_j * \lambda\, d\nu^J$ is nonpositive. After simple transformations, it follows that $\int \check{f}_j * \nu^J\, d\lambda \le 0$, where $\check{f}_j$ is the function defined by $\check{f}_j(x) = f_j(-x)$. As the previous inequality is valid over all nonnegative Radon measure λ with support in $\mathbb{R}^+$, $\check{f}_j * \nu^J$ is nonpositive on $\mathbb{R}^+$.

Step 4: *The Laplace transform.*
Let $r \in \mathbb{R}^+$, and consider the integral $\int e^{-rt}\, d\nu^J(t)$ if it exists. This integral is called the Laplace transform of ν^J at r and is denoted by $L(\nu^J)(r)$. It is well known (see, for instance, Widder 1946, Chap. 2, p. 37) that if this integral converges for some α then there exists an $r_J \le \alpha$ (may be equal to $-\infty$) such that the integral converges for $r > r_J$ and diverges for $r < r_J$. Such an r_J is called the abscissa of convergence of $L(\nu^J)$. Furthermore, if the Radon measure is nonnegative and nonzero, and if r_J is finite, then the limit of the Laplace transform when r converges to r_J from above is equal to infinity.

We begin this step by proving that the abscissa of convergence of ν^J exists, is finite, and is in fact contained in a given bounded interval independent of J. The existence of the abscissa of convergence is a direct consequence of the inequality $\int_0^t d\nu^J(x) \le \rho^t$, which can be proven as in the previous step, but it is now true for all t because the support of ν^J is now equal to all $\mathbb{R}^+$. Now considering $\bar{r} = 3ln\rho/\varepsilon$ ($\bar{r}$ depends only on the investments), it is easy to show that the Laplace transform of ν^J admits an abscissa of convergence called r_J, where r_J is lower or equal to $\bar{r}$, for all finite subsets J of I containing l. Next, we want to show that there exists $\underline{r}$ such that, for all finite subsets J of I, r_J is greater or equal to $\underline{r}$. Using the Fubini theorem and simple transformations, it follows immediately that for all finite subsets J of I, for all j in J, and for all $p > r_J$,

$$\int_0^\infty e^{-pt} \check{f}_j * \nu^J(t)\, dt = \int_0^{+\infty} e^{-pu}\, d\nu^J(u) \int_0^{\tilde{T}_j} e^{pv} f_j(v)\, dv - \int_0^{\tilde{T}_j} e^{-pu}\, d\nu^J(u) \int_u^{\tilde{T}_j} e^{pv} f_j(v)\, dv. \tag{3.1}$$

Recalling that f_k is positive on $]0, \varepsilon[$, it is straightforward to see that there exists a real number $\underline{r}$, such that, for all p lower than or equal to $\underline{r}$, for all u in $[\varepsilon, \tilde{T}_k]$, the integral

$\int_0^u e^{pv} f_k(v)\,dv$ is positive. Now, suppose that there exists a finite subset J of I, such that $\underline{r}$ is greater than r_J. Following the previous step, we get that the function $\check{f}_k * \nu^J$ is nonpositive on $\mathbb{R}^+$ and consequently the integral $\int_0^\infty e^{-\underline{r}t} \check{f}_k * \nu^J(t)\,dt$ is nonpositive. Since ν^J is a nonnegative measure such that $\operatorname{supp}(\nu^J) \cap [0, \tilde{T}_\ell]$ is not empty, and $\tilde{T}_l \leq \tilde{T}_k$, we get that,

$$\begin{aligned}\int_0^{+\infty} e^{-\underline{r}u}\,d\nu^J(u) \int_0^{\tilde{T}_k} e^{\underline{r}v} f_k(v)\,dv \;\geq\; & \int_0^{\tilde{T}_k} e^{-\underline{r}u}\,d\nu^J(u) \int_0^u e^{\underline{r}v} f_k(v)\,dv \\ & + \int_0^{\tilde{T}_k} e^{-\underline{r}u}\,d\nu^J(u) \int_u^{\tilde{T}_k} e^{\underline{r}v} f_k(v)\,dv \\ > & \int_0^{\tilde{T}_k} e^{-\underline{r}u}\,d\nu^J(u) \int_u^{\tilde{T}_k} e^{\underline{r}v} f_k(v)\,dv.\end{aligned}$$

Then, if we apply equation (3.1) to the rate $p = \underline{r}$, and the investment $j = k$, we get a contradiction. Finally, for all finite subsets J of I, r_J is in $[\underline{r}, \bar{r}]$.

We have already seen that the left-hand side of equation (3.1) is nonpositive. In the right-hand side, the first term is the product of $L(f_j)(-p)$ and $L(\nu^J)(p)$. Then, if we take the limit in this term when p converges to r_J, $L(f_j)(-p)$ converges to $L(f_j)(-r_J)$ (recall that f_j has a compact support and then $L(f_j)$ is a continuous function on $\mathbb{R}$) and $L(\nu^J)(p)$ converges to $+\infty$ (recall that ν^J is a nonnegative and nonzero measure and that r_J is the abscissa of convergence of its Laplace transform). It is straightforward that the last term of the right-hand side of equation (3.1) remains bounded when p goes to r_J. Consequently, we have necessarily that $L(f_j)(-r_J)$ is nonpositive. If we recall that $f_j = \mu_j * g$, by a classical property of the Laplace transform, we find that $L(\mu_j)(-r_J)L(g)(-r_J) = L(f_j)(-r_J)$ is nonpositive, which implies that $L(\mu_j)(-r_J)$ is nonpositive for all j in J.

Step 5: *End of the proof of the first implication.*
In the previous step, we proved that for all finite subsets J in I containing investments k and l, there exists a real number r_J contained in $[\underline{r}, \bar{r}]$ such that for all investments j in J, $L(\mu_j)(-r_J)$ is nonpositive. For i in I, let us consider the set $M_i = \{r \in [\underline{r}, \bar{r}] : L(\mu_i)(-r) > 0\}$. Because $L(\mu_i)$ is continuous, M_i is an open set. If $[\underline{r}, \bar{r}] = \cup_{i\in I} M_i$ then there exists a finite subset J of I, such that $[\underline{r}, \bar{r}] = \cup_{j\in J} M_j$. If J does not contain investments k or l we can add them. For this subset J of I and for all r in $[\underline{r}, \bar{r}]$, there exists j in J such that r is in M_j. This contradicts the existence of r_J in $[\underline{r}, \bar{r}]$ (recall that $L(\mu_j)(-r_J)$ is nonpositive for all j in J). Consequently, $[\underline{r}, \bar{r}]$ is not equal to $\cup_{i\in I} M_i$ and there exists r in $[-\bar{r}, -\underline{r}]$ such that for all i in I we have $\int e^{-rt} d\mu_i(t)$ nonpositive.

Step 6: *Proof of the converse implication.*
Assume that there exists a rate r such that, for all i in I, $\int e^{-rt}\,d\mu_i(t)$ is nonpositive and such that the family of investments $(\mu_i)_{i\in I}$ admits an arbitrage. Then, there exists a finite subset J of I and a strategy $(\lambda_j)_{j\in J}$ with investment horizon n, such that the payoff $\pi = \sum_{j\in J} \lambda_j * \mu_j$ is a nonnegative and nonzero measure. Let φ be a nonnegative continuous function with compact support, equal to e^{-rt} on $[0, n]$; we have $\sum_{j\in J} \lambda_j * \mu_j(\varphi) = \pi(\varphi) = L(\pi)(r)$. Since π is a nonnegative and nonzero Radon measure, $L(\pi)$ is positive. Furthermore, $\sum_{j\in J} L(\lambda_j * \mu_j)(r) = \sum_{j\in J} L(\mu_j)(r)L(\lambda_j)(r)$ and each term of this sum is the product of a nonpositive term with a nonnegative one. This contradiction proves the absence of arbitrage opportunities. □

4. APPLICATIONS AND EXAMPLES

4.1. The Discrete Case

In this section, we assume that the cash flows are discrete and that the set of investments I is finite. We will prove that Assumption 3.1 is meaningless.

An investment is denoted by $m_i = (m_0^i, \ldots, m_{T_i}^i)$, and a strategy by $\lambda_i = (\lambda_0^i, \ldots, \lambda_{n-T_i}^i)$, where n is the investment horizon. For each investment m^i, we define the polynomial $P^i(\alpha) = m_0^i + \cdots + m_{T_i}^i \alpha^{T_i}$.

THEOREM 4.1 (Cantor and Lippman 1983; Adler and Gale, 1997). *The absence of arbitrage opportunities is equivalent to the existence of a positive rate α such that, for all investments i, $P^i(\alpha)$ is nonpositive.*

Theorem 4.1 is also equivalent to the following assertion: It is possible to become arbitrarily rich in a finite time if and only if max P^i is positive on $\mathbb{R}_+^*$.

Proof. We will show that Assumption 3.1 is useless. Indeed, suppose that there are no arbitrage opportunities, and that for all i in I, m_0^i is nonpositive. Then, for α small enough we find that $P^i(\alpha)$ is nonpositive and so there is nothing to prove in Theorem 4.1. If for all i in I, m_0^i is equal to zero then it is sufficient to start at date $t = 1$. Consequently, we can always assume that there exists an investment k such that m_0^k is positive. In the same way, we could show that it is always possible to find an investment ℓ, such that $m_{T_\ell}^\ell$ is positive. Assumption 3.1 is always satisfied.

Using Theorem 3.1, the absence of arbitrage opportunities is equivalent to the existence of a rate r such that $\int e^{-rt}\, d\mu_i(t)$ is nonpositive for all i in I. Here, μ_i is equal to $\sum_{t=0}^{T_i} m_t^i \delta_t$, thus $\int e^{-rt}\, d\mu_i(t) = \sum_{t=0}^{T_i} m_t^i e^{-rt}$, which gives the result with $\alpha = e^{-r}$. □

4.2. Examples

First, consider the case of a "plan d'épargne logement." In this case, and if we simplify the product, it is divided in two stages. During the first stage, the investor saves at a fixed rate r. In the second stage, he can obtain a loan at a special rate r' near to r. The bank receives 1^F today. After one period it returns $(1+r)^F$, and lends 1^F. Finally, at the last period the bank receives $(1+r')^F$. We denote this investment by $m = (1, -2-r, 1+r')$. Our main result is that there is an arbitrage opportunity if, for all positive real numbers x, $1-(2+r)x+(1+r')x^2$ is positive. A simple computation leads to the following condition $r' - r > (r^2/4)$. Considering a rate r of 5.25, it is possible for the bank to construct an arbitrage opportunity if $r' > 5.32$.

Next, consider the case of a borrowing rate and a lending rate which are equal. This situation can be described by the investments $(-1, 1+r)$ and $(1, -1-r)$. Assume that there exists another investment i, then it is straightforward to see that the absence of arbitrage opportunities implies that $P^i(1/(1+r)) \leq 0$.

Other examples are provided in Adler and Gale (1997).

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