Optimal investment with taxes: an optimal control problem with endogeneous delay

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1. Introduction

The simplest consumption–investment problem can be formulated as follows. There is an economic agent with preferences described by a utility function $U(c) = \int_0^T u(t, c(t))\,\mathrm{d}t$, where c is the consumption path in the time interval $[0, T]$. The agent has an income function ω defined on $[0, T]$. The financial market consists in one riskless asset with price function S. At each time t, the agent receives an income $\omega(t)$, rebalances his portfolio (by buying or selling some financial assets) and spends the rest for consumption. Then, Ando and Modigliani [1] proved that the optimal consumption behaviour of the agent is constant over the time interval $[0, T]$.

Here, we study the case where the portfolio rebalancement involves the payment of taxes on benefits. Then, the purchasing time of the asset to be sold has to be recorded in order to compute the amount of tax to be paid. In addition to the no-short-selling constraint, our model assumes that sales are subject to the first-in-first-out priority rule on sales. A precise description of the model is given in Section 2. The agent problem

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turns out to be a nonclassical optimal control problem with *endogeneous delay* and with complex nonnegativity constraint on consumption.

Section 3 is devoted to the proof of the following economic appealing result. An optimal strategy can always be chosen such that the agent never sells out of his portfolio and buy new financial assets simultaneously. Using this property, the nonnegativity constraint on consumption is simplified and reduced to a classical constraint on the controls and the state variables. Namely, the nonnegativity constraint on consumption can be expressed simply in terms of the investment and the disinvestment functions.

In Section 4, we assume some additional smoothness conditions on the optimal strategy in order to derive the first-order conditions associated to the control problem of interest. The usual variational methods are adapted to handle the endogeneous delay function.

The existence problem as well as the numerical computation of the optimal solution using the first-order conditions are left for future work.

2. The model

2.1. The financial market

There is a single consumption commodity available for consumption through $[0,T]$ where T is a finite time horizon. The financial market consists in one riskless asset, called bond, whose price function is given by

$$S(t)=S(0)\exp\int_0^t r(s)\,ds, \quad t\in[0,T],$$

where $r(.)$ is a continuous nonnegative function defined on $[0,T]$; $r(t)$ is the instantaneous interest rate at time t.

2.2. Taxation rule

In this paper, we assume that sales are subject to taxes on benefits.[1] More precisely, we shall consider the usual *first-in-first-out* rule according to which any bond sold at some time t should be the oldest one in the time t portfolio.

We introduce the set $\Delta=\{(t,u)\in R^2\colon\ 0\le u\le t\le T\}$. Fix some (t,u) in Δ. For each monetary unit invested at time u and sold out at time t, we denote by $\varphi(t,u)$ the after tax amount received at time t, i.e. the amount of tax paid by the investor is

$$\frac{S(t)}{S(u)}-\varphi(t,u).$$

The after tax return function φ defined on Δ is assumed to satisfy the following conditions.

[1] Since the instantaneous interest rate is nonnegative, sales always yield some nonnegative benefit.

Assumption 2.1. φ is a C^1 function mapping Δ into $[1,+\infty)$ with $\varphi(t,t)=1$, for all $t\in[0,T]$, and

$$\frac{\varphi_t}{\varphi}(t,.) \quad \text{is decreasing for any } t\in[0,T]. \tag{2.1}$$

The fact that $\varphi\geq 1$ is a natural condition on the after tax return function φ since the asset price $S(t)$ is nondecreasing and the tax is a (possibly varying) proportion of the capital gains. The restriction $\varphi(t,t)=1$ is a natural condition which expresses the fact that there is no benefit from selling and buying the same share of a financial asset at the same time t. The technical condition (2.1) is needed for the proof of Theorem 3.1 which is essential for all our analysis. As, it is illustrated by the following example, this condition is satisfied in most classical taxation cases.

Example 2.1. Constant tax rate. Suppose that the tax to be paid for one asset bought at time u and sold at time t is given by $\tau[S(t)-S(u)]$. Therefore, the investor return from such a strategy is $\varphi(t,u)=[S(t)-\tau(S(t)-S(u))]/S(u)=\tau+(1-\tau)S(t)/S(u)=\tau+(1-\tau)\exp\int_u^t r(s)\,\mathrm{d}s$. It is easily checked that φ satisfies the conditions of Assumption 2.1.

Let us note that condition 2.1, together with the other conditions of Assumption 2.1, is stronger then the following economic appealing one:

$$\varphi(t,u)\varphi(u,v)\leq\varphi(t,v) \quad \text{for all } 0\leq v\leq u\leq t\leq T. \tag{2.2}$$

The latter condition says that one cannot save some taxes by selling and buying the asset S at any intermediate date u between v and t. To see that Eq. (2.2) follows from Assumption 2.1, denote by $\phi(t,u)=\ln(\varphi(t,u))$ and consider the function $f(t)=\phi(t,u)+\phi(u,v)-\phi(t,v)$ defined on $[u,T]$ for fixed (u,v) in Δ. Then since $\varphi(t,t)=1$, we have $f(u)=0$. Furthermore, it is easily checked that $f'(t)\leq 0$ which provides Eq. (2.2).

Remark 2.1. *An immediate consequence of Eq.* (2.2) *and the fact that* φ *is valued in* $[1,\infty)$ *is that* $\varphi(t,u)$ *is nondecreasing in* t *and nonincreasing in* u.

2.3. Trading strategies

We denote by L^1_+ the set of all nonnegative $L^1[0,T]$ functions. Let (x,y) be a pair of L^1_+ functions. Here, $x(t)$ (resp. $y(t)$) is the investment rate (resp. disinvestment) in units of the risky asset at time t. In other words, $\int_0^t x(s)\,\mathrm{d}s$ (resp. $\int_0^t y(s)\,\mathrm{d}s$) is the cumulated number of assets purchased (sold out) up to time t. Such a pair (x,y) is said to be a trading strategy if the no short selling constraint

$$\int_0^t y(s)\,\mathrm{d}s\leq\int_0^t x(s)\,\mathrm{d}s, \quad 0\leq t\leq T \tag{2.3}$$

holds. Condition (2.3) says that sales must not exceed purchases at any time. Given a trading strategy (x, y), we define the delay function $\theta^{x,y}$ by

$$\theta^{x,y}(t) = \sup\left\{ s \in [0,t]\colon \int_0^s x(u)\,\mathrm{d}u \leq \int_0^t y(u)\,\mathrm{d}u \right\}.$$

In the sequel, we shall write θ for $\theta^{x,y}$ for notation simplification. As defined, θ is nondecreasing and whenever $\int_0^t y(s)\,\mathrm{d}s > 0$, $\theta(t)$ is the purchasing date of the last asset sold out from the portfolio. If $\int_0^t x(s)\,\mathrm{d}s = \int_0^t y(s)\,\mathrm{d}s = 0$ (no market participation up to time t a.e.), then $\theta(s) = s$ for all $s \in [0, t]$. Furthermore, from the no short sales constraint (2.3), we have

$$\theta(0) = 0 \leq \theta(t) \leq t, \quad 0 \leq t \leq T. \tag{2.4}$$

Remark 2.2. *Since x and y are integrable (Lebesgue), the functions*

$$t \mapsto \int_0^t x(s)\,\mathrm{d}s \quad \textit{and} \quad t \mapsto \int_0^t y(s)\,\mathrm{d}s$$

belong to the Sobolev space $W^{1,1}[0,T]$ which coincides with the set of all absolutely continuous functions, see e.g. Brézis [3], *Remark* 8, *p.* 125.

We then have the following useful properties of θ.

Lemma 2.1. (i) *θ is right-continuous on* $[0,T]$, *i.e.* $\theta(t^+) = \theta(t)$ *for all* $t \in [0,T]$,
(ii) *for all* $t \in [0,T]$, $\int_0^{\theta(t)} x(s)\,\mathrm{d}s = \int_0^t y(s)\,\mathrm{d}s$,
(iii) *for all* $t \in [0,T]$, $\int_{\theta(t^-)}^{\theta(t)} x(s)\,\mathrm{d}s = 0$.

Proof. (i) Fix some $t \in [0, T)$. By definition of θ, we have that $\int_0^{\theta(t+\varepsilon)} x(s)\,\mathrm{d}s \leq \int_0^{t+\varepsilon} y(s)\,\mathrm{d}s$ for all $\varepsilon > 0$. Since $y \in L^1_+$, this provides

$$\int_0^{\theta(t^+)} x(s)\,\mathrm{d}s \leq \int_0^t y(s)\,\mathrm{d}s,$$

see Remark 2.2. Now, since $\theta(t^+) \leq t$ (because $\theta(t+\varepsilon) \leq t+\varepsilon$), this proves that $\theta(t^+) \leq \theta(t)$, by definition of θ, and therefore $\theta(t+) = \theta(t)$ since θ is nondecreasing.

(ii) First, suppose that $\theta(t) < t$. Then, by definition of θ, we have $\int_0^{\theta(t)} x(s)\,\mathrm{d}s \leq \int_0^t y(s)\,\mathrm{d}s \leq \int_0^{\theta(t)+\varepsilon} x(s)\,\mathrm{d}s$ for sufficiently small $\varepsilon > 0$. The required result is obtained by letting ε go to zero, recall that $x \in L^1_+$ and use Remark 2.2. If $\theta(t) = t$, then, from the no short sales constraint and the nonnegativity of y, we have

$$\int_0^t y(s)\,\mathrm{d}s \leq \int_0^t x(s)\,\mathrm{d}s = \int_0^{\theta(t)} x(s)\,\mathrm{d}s \leq \int_0^t y(s)\,\mathrm{d}s.$$

(iii) From the fact that x and y are in L^1_+, it is easily checked that $\int_0^{\theta(t^+)} x(s)\,\mathrm{d}s = \int_0^{\theta(t^-)} x(s) = \int_0^t y(s)\,\mathrm{d}s$, see Remark 2.2. □

2.4. The agent problem

At each time $t \in [0,T]$, the agent is endowed with an income rate $\omega(t)$ in units of the consumption good. Here ω is a given positive continuous function on $[0,T]$. Then, given a trading strategy (x,y), the agent consumption rate function is given by

$$c^{x,y}(t) = \omega(t) - x(t)S(t) + y(t)\varphi(t,\theta^{x,y}(t))S(\theta^{x,y}(t)), \quad t \in [0,T]. \tag{2.5}$$

Therefore, a trading strategy (x,y) is said to be feasible if the induced consumption rate function is nonnegative.

The agent preferences are represented by a time-additive utility function from consumption $U(t,c)$. We assume throughout the paper that U is $C^{1,2}([0,T],\mathbb{R}_+)$, decreasing in t and concave nondecreasing in c.

We are now able to write the control problem of the agent. An admissible trading strategy (x,y) is a feasible trading strategy such that

$$\int_0^T |U(t,c^{x,y}(t))|\,\mathrm{d}t < \infty. \tag{2.6}$$

We shall denote by $\mathscr{A}$ the set of all admissible trading strategies, i.e.

$$\mathscr{A} = \{(x,y) \in L^1_+ \times L^1_+ : \text{ Eqs. (2.3) and (2.6) hold and } c^{x,y} \geq 0\}. \tag{2.7}$$

The agent optimal control problem is

$$\sup_{(x,y)\in\mathscr{A}} \int_0^T U(t,c^{x,y}(t))\,\mathrm{d}t, \tag{2.8}$$

i.e. maximize the utility from consumption over all admissible trading strategies. Here, we derive the first-order conditions corresponding to an optimum. The optimal control problem (2.8) is nonstandard for two reasons:

(i) the presence of the delay function $\theta^{x,y}$ in the expression of the consumption rate function (2.5),

(ii) the constraint $c^{x,y} \geq 0$ which involves x, y and $\theta^{x,y}$.

Our approach is the following. In Section 3, we prove that it is not optimal to purchase new assets and to sell out from the portfolio at the same time t (in the a.e. sense). We shall see that the condition given by Eq. (2.2) does not seem to be sufficient for the derivation of this property. This economic appealing result is essential in order to simplify the nonnegativity constraint on consumption and to reduce the control problem to an optimal control problem with endogeneous delay, but with more standard constraints. In Section 4, assuming some smoothness conditions on x and y, we derive the first order conditions by adapting the classical variational approach which allows to derive the Pontryagin principle.

3. Simplifying the nonnegativity constraint on consumption

In this section, we prove the following basic result.

Theorem 3.1. *Let* (x, y) *be some admissible strategy in* $\mathscr{A}$. *Then, there exists an admissible strategy* $(\tilde{x}, \tilde{y}) \in \mathscr{A}$ *such that* $c^{x,y} \leq c^{\tilde{x},\tilde{y}}$ *and*

$$\tilde{x}(t)\tilde{y}(t) = 0, \quad 0 \leq t \leq T \ \text{a.e.} \tag{3.1}$$

The last theorem has an appealing economic interpretation: if there exists an optimal strategy for the optimal consumption investment problem, then it can be chosen such that the agent never sells out from his portfolio and buys new assets simultaneously.

Before turning to the proof of the last theorem, we state an important consequence of it in terms of constraints simplification. Theorem 3.1 says that the set of admissible strategies in the optimization problem (2.8) can be restricted to the subset $\mathscr{A}_0$ of $\mathscr{A}$ whose elements satisfy condition (3.1). In view of the expression of the consumption rate function in Eq. (2.5), it follows that, for all $(x, y) \in \mathscr{A}_0$, $c^{x,y}(t) < 0$ if and only if $x(t)S(t) > \omega(t)$ (since $y(t)\varphi(t, \theta(t))S(\theta(t)) \geq 0$). We then have the following result.

Corollary 3.1. *Let* $\mathscr{A}_0$ *be the subset of* $\mathscr{A}$ *given by*

$$\mathscr{A}_0 = \{(x, y) \in L^1_+ \times L^1_+ : xS \leq \omega, \ \textit{Eqs.}\ (2.3),\ (2.6)\ \textit{and}\ (3.1)\ \textit{hold}\}.$$

Then, we have

$$\sup_{(x,y)\in\mathscr{A}} \int_0^T U(t, c^{x,y}(t))\,\mathrm{d}t = \sup_{(x,y)\in\mathscr{A}_0} \int_0^T U(t, c^{x,y}(t))\,\mathrm{d}t.$$

The rest of this section is devoted to the long proof of Theorem 3.1.

3.1. The case of a positive investment rate function

We first prove Theorem 3.1 when the investment rate function x is such that

$$x(t) \geq \delta, \quad 0 \leq t \leq T \ \text{a.e.}$$

for some $\delta > 0$. This is a preliminary result for the proof of Theorem 3.1 for general investment functions x.

Let A be a Borel subset of $[0, T]$ with positive Lebesgue measure and suppose that

$$y(t) > 0 \quad \text{for all } t \in A.$$

Then define

$$x^{\varepsilon}(t) = x(t) - \varepsilon \frac{h(t)}{S(t)}, \quad 0 \leq t \leq T, \tag{3.2}$$

where h is an L^1_+ function with $h > 0$ on A and $h = 0$ on A^c. We intend to prove the existence of some y^{ε} such that

$$(x^{\varepsilon}, y^{\varepsilon}) \in \mathscr{A} \quad \text{and} \quad c^{x,y}(t) = c^{x^{\varepsilon}, y^{\varepsilon}}(t), \quad 0 \leq t \leq T \ \text{a.e.}$$

i.e.

$$y^{\varepsilon}(t) f(t, \theta^{\varepsilon}(t)) = y(t) f(t, \theta(t)) - \varepsilon h(t), \quad 0 \leq t \leq T, \tag{3.3}$$

where $\theta^\varepsilon = \theta^{x^\varepsilon, y^\varepsilon}$ and

$$f(t,u) = \varphi(t,u)S(u) \quad \text{for all } (t,u) \in \Delta.$$

Remark 3.1. *In order to ensure that x^ε and y^ε are in L^1_+, we can think of the particular choice of h given by*

$$h(t) = \tfrac{1}{2}\min\{x(t)S(t), y(t)f(t,\theta(t))\}1_A, \quad 0 \le t \le T.$$

Moreover, if we restrict ε to the interval $[0,1]$, then the perturbed investment rate function x^ε remains bounded from below by the constant $\delta/2$.

Remark 3.2. *When the investment rate function is known to be positive a.e. the delay function is continuous, see Lemma* 2.1(iii).

In the rest of this paragraph, we intend to prove that equation (3.3) admits a solution y^ε such that $(x^\varepsilon, y^\varepsilon)$ is an admissible trading strategy. From Remark 3.1 and the definition of y^ε as a solution to Eq. (3.3) (if exists), we have to ensure, in particular, that the no short sales condition (2.3) holds for $(x^\varepsilon, y^\varepsilon)$.

For fixed $(t,\varepsilon) \in [0,T] \times [0,1]$, we introduce the function $\phi(t,.,\varepsilon)$ defined on $[0,T]$ by

$$\phi(t,\zeta,\varepsilon) = \int_0^\zeta x^\varepsilon(u) f(t,u)\,\mathrm{d}u, \quad 0 \le \zeta \le T.$$

Then, since $x^\varepsilon(.)f(t,.)$ is a positive $L^1[0,T]$ function, $\phi(t,.,\varepsilon)$ is an increasing continuous function for all fixed (t,ε). Therefore, it admits a $[0,T]$-valued continuous increasing inverse function $\psi(t,.,\varepsilon)$ defined by

$$z = \phi(t,\zeta,\varepsilon) \Leftrightarrow \zeta = \psi(t,z,\varepsilon).$$

Remark 3.3. *Since x^ε is nonincreasing in ε, the function $\psi(t,z,.)$ is nondecreasing.*

Remark 3.4. *For fixed (t,z), the function $\psi(t,z,.)$ is continuous on $[0,1]$. To see this, consider an arbitrary small parameter $\eta \neq 0$. From the obvious equality $\phi(t,\psi(t,z,\varepsilon), \varepsilon) = \phi(t,\psi(t,z,\varepsilon+\eta),\varepsilon+\eta)$, we see that*

$$\int_{\psi(t,z,\varepsilon+\eta)}^{\psi(t,z,\varepsilon)} f(t,u)x^\varepsilon(u)\,\mathrm{d}u + \eta\int_0^{\psi(t,z,\varepsilon+\eta)} \varphi(t,u)h(u)\,\mathrm{d}u = 0.$$

Now, from Remark 3.3, the limits $\psi(t,z,\varepsilon^+)$ and $\psi(t,z,\varepsilon^-)$ exist and are in $[0,T]$. Then, sending η to zero in the last equality provides

$$\int_{\psi(t,z,\varepsilon^+)}^{\psi(t,z,\varepsilon)} f(t,u)x^\varepsilon(u)\,\mathrm{d}u = \int_{\psi(t,z,\varepsilon^-)}^{\psi(t,z,\varepsilon)} f(t,u)x^\varepsilon(u)\,\mathrm{d}u = 0,$$

which provides the required result from the positivity of fx^ε.

Remark 3.5. *By a similar argument to Remark* 3.4, *it is easily checked that* $\psi(.,z,\varepsilon)$ *is continuous on* $[0,T]$ *for any fixed* (z,ε).

Proposition 3.1. *Let* $\varepsilon\in[0,1]$. *Suppose that there exists a continuous solution* z^ε *to the integral equation*

$$z(t)=\int_0^t [\phi_t(u,\psi(u,z(u),\varepsilon),\varepsilon)+y(u)f(u,\theta(u))-\varepsilon h(u)]\,\mathrm{d}u, \tag{3.4}$$

$0\le t\le T$ *(with* $\theta=\theta^{x,y}$*), satisfying*

$$\psi(t,z^\varepsilon(t),\varepsilon)\le t \quad \text{for all } t\in[0,T].$$

Then, there exists a solution y^ε *to Eq.* (3.3) *such that* $(x^\varepsilon,y^\varepsilon)$ *is an admissible trading strategy. Moreover, we have*

$$z^\varepsilon(t)=\phi(t,\theta^{x^\varepsilon,y^\varepsilon}(t),\varepsilon),\quad 0\le t\le T.$$

Proof. Let z^ε be a continuous solution of Eq. (3.4) and assume that the function ζ^ε defined by $\zeta^\varepsilon(t)=\psi(t,z^\varepsilon(t),\varepsilon)$ satisfies $\zeta^\varepsilon(t)\le t$ for all $t\in[0,T]$. From Eq. (3.4), it is easily seen that z^ε is nondecreasing and nonnegative. This implies that ζ^ε is a continuous nondecreasing and nonnegative function, see Remark 3.5.

By the dominated convergence Theorem, it is easily checked that ϕ is differentiable with respect to the variable t and we have $\phi_t(t,\zeta^\varepsilon(t),\varepsilon)=\int_0^{\zeta^\varepsilon(t)}x^\varepsilon(u)f_t(t,u)\,\mathrm{d}u$ for all $t\in[0,T]$. Then, by definition of ϕ and ψ, we get

$$\int_0^t \phi_t(u,\psi(u,z^\varepsilon(u),\varepsilon),\varepsilon)\,\mathrm{d}u=\int_0^t\left(\int_0^{\zeta^\varepsilon(u)}x^\varepsilon(s)f_t(u,s)\,\mathrm{d}s\right)\mathrm{d}u,$$

which provides by the Fubini theorem:

$$\begin{aligned}\int_0^t \phi_t(u,\psi(u,z^\varepsilon(u),\varepsilon),\varepsilon)\,\mathrm{d}u&=\int_0^{\zeta^\varepsilon(t)}x^\varepsilon(s)\int_{(\zeta^\varepsilon)^{-1}(s)}^t f_t(u,s)\,\mathrm{d}u\,\mathrm{d}s\\&=z^\varepsilon(t)-\int_0^{\zeta^\varepsilon(t)}f((\zeta^\varepsilon)^{-1}(s),s)x^\varepsilon(s)\,\mathrm{d}s,\end{aligned} \tag{3.5}$$

where $(\zeta^\varepsilon)^{-1}$ is the right-continuous inverse function of the nondecreasing continuous function ζ^ε. We now use the following result whose proof will be carried out later.

Lemma 3.2. *The function* $t\mapsto\int_0^{\zeta^\varepsilon(t)}x^\varepsilon(u)\,\mathrm{d}u$ *is absolutely continuous.*

We then define y^ε as the L^1_+ generalized derivative of the nondecreasing function $t\mapsto\int_0^{\zeta^\varepsilon(t)}x^\varepsilon(u)\,\mathrm{d}u$. By definition, we have

$$\int_0^s y^\varepsilon(u)\,\mathrm{d}u=\int_0^{\zeta^\varepsilon(s)}x^\varepsilon(u)\,\mathrm{d}u,\quad 0\le s\le T. \tag{3.6}$$

Note that from the positivity of x^ε and the fact that $\zeta^\varepsilon(t) \le t$, we have $\zeta^\varepsilon = \theta^{x^\varepsilon, y^\varepsilon} = \theta^\varepsilon$. Now, the integral on the right-hand side of Eq. (3.6) can be seen as a Stieljes integral with respect to the measure $d(\int_0^s x^\varepsilon(u)\,\mathrm{d}u)$. Then, from Eq. (3.6), the change of variable formula for Stieljes integrals leads to

$$\int_0^t \phi_t(u, \psi(u, \theta^\varepsilon(u), \varepsilon), \varepsilon)\,\mathrm{d}u = z^\varepsilon(t) - \int_0^t f(s, \theta^\varepsilon(s)) y^\varepsilon(s)\,\mathrm{d}s$$

since $\theta^\varepsilon(0) = 0$, see e.g. Riesz and Nagy [7]. Since z^ε is a solution to the integral equation (3.4), the last equality provides:

$$\int_0^t f(s, \theta^\varepsilon(s)) y^\varepsilon(s)\,\mathrm{d}s = \int_0^t [f(s, \theta(s)) y(s) - \varepsilon h(s)]\,\mathrm{d}s, \quad 0 \le t \le T.$$

Since the terms inside the integrals are in $L^1[0, T]$, the left-hand side term as well as the right-hand side one, as functions of t, are absolutely continuous. This proves that y^ε solves (3.3) by uniqueness of the generalized derivative.

Proof. Since z^ε is a continuous solution of the integral equation (3.4), it is easy to see that it is absolutely continuous. Now, take an arbitrary $\alpha > 0$. Then there exists $\eta > 0$ such that for any family of nonintersecting intervals $\{(t_i, t_i'),\ i = 1, \ldots, n\}$, satisfying $\sum_i |t_i' - t_i| < \eta$, we have

$$\sum_i |z^\varepsilon(t_i') - z^\varepsilon(t_i)| < \alpha.$$

Next, let $\zeta_i = \zeta^\varepsilon(t_i) = \psi(t_i, z^\varepsilon(t_i), \varepsilon)$ and $\zeta_i' = \zeta^\varepsilon(t_i') = \psi(t_i', z^\varepsilon(t_i'), \varepsilon)$. By the definition of ψ, we have

$$\sum_i |\phi(t_i, \zeta_i, \varepsilon) - \phi(t_i', \zeta_i', \varepsilon)| < \alpha.$$

This provides

$$\sum_i \left| \int_{\zeta_i}^{\zeta_i'} x^\varepsilon(u) f(t_i, u)\,\mathrm{d}u \right| < \alpha + \sum_i \left| \int_0^{\zeta_i'} x^\varepsilon(u)[f(t_i', u) - f(t_i, u)]\,\mathrm{d}u \right|,$$

by the triangular inequality, which implies that

$$\begin{aligned} \tfrac{1}{2}\delta S(0) \sum_i |\zeta_i - \zeta_i'| &< \alpha + \|x^\varepsilon\|_1 \|f_t\|_\infty \sum_i |t_i' - t_i| \\ &< \alpha + \|x^\varepsilon\|_1 \|f_t\|_\infty \eta, \end{aligned}$$

where δ is the lower bound on x. This proves that ζ^ε is absolutely continuous and therefore the required result follows from the fact that $x^\varepsilon \in L^1$. $\square$

In the sequel, we introduce the notation

$$L(t, z, \varepsilon) = \phi_t\,(t, \psi(t, z, \varepsilon), \varepsilon) + y(t) f(t, \theta(t)) - \varepsilon h(t).$$

The integral equation (3.4) can then be rewritten in

$$z(t) = \int_0^t L(u, z(u), \varepsilon)\, \mathrm{d}u, \quad 0 \le t \le T. \tag{3.7}$$

Remark 3.6. *By elementary differential calculus, it is easily checked that, for any fixed* $t \in [0, T]$, *the function* $L(t,.,.)$ *is* C^1 *and*

$$\frac{\partial L}{\partial z}(t, z, \varepsilon) = \frac{\varphi_t}{\varphi}\, (t, \psi(t, z, \varepsilon)), \tag{3.8}$$

$$\frac{\partial L}{\partial \varepsilon}(t, z, \varepsilon) = \frac{\varphi_t}{\varphi}(t, \psi(t, z, \varepsilon)) \int_0^{\psi(t,z,\varepsilon)} \varphi(t, u) h(u)\, \mathrm{d}u$$

$$- \int_0^{\psi(t,z,\varepsilon)} \varphi_t(t, u) h(u)\, \mathrm{d}u - \int_0^t h(u)\, \mathrm{d}u, \tag{3.9}$$

since $x^\varepsilon \ge \delta/2 > 0$.

Lemma 3.3. *For any* $\varepsilon \in [0, 1]$, *the integral equation* (3.7) *has a unique continuous solution* z^ε.

Proof. Denote by $\mathscr{X} = C[0, T]$ the set of all real-valued functions defined and continuous on $[0, T]$. The set $\mathscr{X}$ is a Banach space when equipped with the norm $\|.\|_\infty$. We define the operator H^ε on $\mathscr{X}$ by

$$H^\varepsilon g(t) = \int_0^t L(u, g(u), \varepsilon)\, \mathrm{d}u.$$

It is clear that $H^\varepsilon g \in \mathscr{X}$. We intend to use a contraction argument on H^ε in the set $\mathscr{X}$ which will prove the existence of a unique fixed point for H^ε in $\mathscr{X}$. Take two arbitrary elements $g^{(1)}$, $g^{(2)}$ in $\mathscr{X}$. Then from Eq. (3.8), it is easily checked that

$$\sup_{0 \le u \le t} |H^\varepsilon g^{(1)}(u) - H^\varepsilon g^{(2)}(u)| \le t \left\| \frac{f_t}{f} \right\|_\infty \sup_{0 \le u \le t} |g^{(1)}(u) - g^{(2)}(u)|.$$

The last inequality shows that H^ε is a contraction on $\mathscr{X}$ for a sufficiently small t. Therefore, the required result is obtained by reasonning locally in t. $\square$

In order to prove the existence of a solution y^ε to Eq. (3.3), such that $(x^\varepsilon, y^\varepsilon) \in \mathscr{A}$, it remains to prove that $\zeta^\varepsilon(t) \le t$, i.e. $\psi(t, z^\varepsilon(t), \varepsilon) \le t$, for all $t \in [0, T]$, as required by Lemma 3.3. In the rest of this paragraph, we shall prove that $\zeta^\varepsilon \le \theta$ which suffices to prove the required result.

Remark 3.7. *Recall that* $z^\varepsilon(t) = \int_0^{\zeta^\varepsilon(t)} x^\varepsilon(u) f(t, u)\, \mathrm{d}u$ *and therefore*

$$\int_0^{\zeta^\varepsilon(t)} x(u) f(t, u)\, \mathrm{d}u = z^\varepsilon(t) + \varepsilon \int_0^{\zeta^\varepsilon(t)} h(u) \varphi(t, u)\, \mathrm{d}u.$$

Therefore, in order for $\zeta^\varepsilon(t)$ to be nonincreasing in ε in a neighbourhood of $\varepsilon = 0$ it suffices that

$$\left[\frac{\partial z^\varepsilon}{\partial \varepsilon}(t) + \int_0^{\zeta^\varepsilon(t)} h(u)\varphi(t,u)\,\mathrm{d}u\right]_{\varepsilon=0} < 0,$$

whenever $\partial z^\varepsilon/\partial\varepsilon$ and $\partial\zeta^\varepsilon/\partial\varepsilon$ exist.

Remark 3.8. *The unique solution z^ε of the integral equation* (3.7) *is nonincreasing in ε. To see this, let $g(t,\varepsilon)$ be a real-valued function continuous in $t \in [0,T]$ and C^1 nonincreasing in ε. Define the operator H by $Hg(t,\varepsilon) = H^\varepsilon g^\varepsilon(t)$ where $g^\varepsilon(t) = g(t,\varepsilon)$. Then Hg is C^1 in ε and*

$$\frac{\partial Hg}{\partial \varepsilon}(t,\varepsilon) = \int_0^t L_\varepsilon(u, g(u,\varepsilon), \varepsilon)\,\mathrm{d}u + \int_0^t L_z(u, g(u,\varepsilon), \varepsilon)\frac{\partial g}{\partial \varepsilon}(u,\varepsilon)\,\mathrm{d}u.$$

The second term on the right-hand side is nonpositive since g is nonincreasing in ε, see Eq. (3.8) *and Remark Eq.* 2.1. *In view of Eq.* (3.9), *the other term can be written in*

$$\int_0^t L_\varepsilon(u, g(u,\varepsilon), \varepsilon)\,\mathrm{d}u = -\int_0^t h(u)\,\mathrm{d}u + \int_0^{\psi(t,g(t,\varepsilon),\varepsilon)} f(t,u)h(u) \times \left[\frac{f_t}{f}(t, \psi(t, g(t,\varepsilon), \varepsilon)) - \frac{f_t}{f}(t,u)\right]\mathrm{d}u \le 0,$$

since $(f_t/f)(t,.)$ is nonincreasing by definition of the after tax return function φ. Therefore, recalling that z^ε is a fixed point of the operator H^ε, we can conclude that the function $\varepsilon \mapsto z^\varepsilon$ is nonincreasing; the C^1 property is not necessarily inherited at the limit. Unfortunately, this is not sufficient to prove that $\zeta^\varepsilon(t)$ is nonincreasing in ε, see Remark 3.7.

Lemma 3.4. *For any fixed t in $[0,T]$, the function $\varepsilon \mapsto z^\varepsilon(t)$ lies in the Sobolev space $W^{1,\infty}$.*

Proof. Let ε and ε' in $[0,1)$ and denote by $\delta(t) = \left|z^\varepsilon(t) - z^{\varepsilon'}(t)\right|$. Then, since L_z and L_ε are bounded, see Eqs. (3.8) and (3.9), the integral equation (3.7) provides

$$\delta(t) \le M\left(|\varepsilon - \varepsilon'| + \int_0^t \delta(u)\,\mathrm{d}u\right), \quad 0 \le t \le T$$

for some constant M. From the Gronwall inequality, this proves that $\delta(t) \le M'|\varepsilon - \varepsilon'|$ for some constant M'. The required result follows from Brézis [3], Corollary 4, p. 126. □

To simplify the notations, we introduce the generalized derivative of z^{ε} with respect to ε:

$$\tilde{g}^{\varepsilon}(t)=\frac{\partial z^{\varepsilon}}{\partial \varepsilon}(t), \quad 0\leq t\leq T,\ 0\leq \varepsilon\leq 1.$$

Then, $\tilde{g}^{\varepsilon}$ solves the following linear integral equation

$$\tilde{g}^{\varepsilon}(t)=\int_0^t [L_z(u,z^{\varepsilon}(u),\varepsilon)\tilde{g}^{\varepsilon}(u)+L_{\varepsilon}(u,z^{\varepsilon}(u),\varepsilon)]\,\mathrm{d}u,$$

see Remark 3.6 together with Brézis [3], Corollary 10, p. 131. This provides

$$\begin{aligned}
\tilde{g}^{\varepsilon}(t) &- \int_0^t \frac{\varphi_t}{\varphi}(u,\zeta^{\varepsilon}(u))\tilde{g}^{\varepsilon}(u)\,\mathrm{d}u \\
&= -\int_0^t h(u)\,\mathrm{d}u - \int_0^t \int_0^{\zeta^{\varepsilon}(u)} \varphi_t(u,v)h(v)\,\mathrm{d}v\,\mathrm{d}u \\
&\quad + \int_0^t \frac{\varphi_t}{\varphi}(u,\zeta^{\varepsilon}(u)) \int_0^{\zeta^{\varepsilon}(u)} \varphi(u,v)h(v)\,\mathrm{d}v\,\mathrm{d}u.
\end{aligned} \tag{3.10}$$

Next, following Remark 3.7, we introduce the function g^{ε} defined by

$$g^{\varepsilon}(t)=\tilde{g}^{\varepsilon}(t)+\int_0^{\zeta^{\varepsilon}(t)} \varphi(t,u)h(u)\,\mathrm{d}u, \quad (t,\varepsilon)\in[0,T]\times[0,1]. \tag{3.11}$$

Lemma 3.5. *The function* $(t,\varepsilon)\mapsto g^{\varepsilon}(t)$ *is continuous on* $[0,T]\times[0,1]$ *and, for all* $t\in[0,T]$, $g^0(t)<0$ *whenever* $\int_0^t h(u)\,\mathrm{d}u>0$.

Proof. From Eq. (3.10) and the definition of g^{ε} in Eq. (3.11), we see that g^{ε} solves the linear integral equation:

$$g^{\varepsilon}(t)-\int_0^t \frac{\varphi_t}{\varphi}(u,\zeta^{\varepsilon}(u))g^{\varepsilon}(u)\,\mathrm{d}u=H(t),$$

where

$$H(t)=\int_0^{\zeta^{\varepsilon}(t)} \varphi((\zeta^{\varepsilon})^{-1}(u),u)h(u)\,\mathrm{d}u-\int_0^t h(u)\,\mathrm{d}u.$$

Then, it is easily checked that the solution of this first-order differential equation is given by

$$g^{\varepsilon}(t)=H(t)+\int_0^t H(u)\frac{\varphi_t}{\varphi}(u,\zeta^{\varepsilon}(u))\exp\left(\int_u^t \frac{\varphi_t}{\varphi}(v,\zeta^{\varepsilon}(v))\,\mathrm{d}v\right)\mathrm{d}u.$$

Substituting the expression of H and applying the Fubini theorem provides

$$g^{\varepsilon}(t) = -\int_0^t h(u)\exp\left(\int_u^t \frac{\varphi_t}{\varphi}(v,\zeta^{\varepsilon}(v))\,\mathrm{d}v\right)\mathrm{d}u$$

$$\int_0^{\zeta^{\varepsilon}(t)} h(u)\varphi((\zeta^{\varepsilon})^{-1}(u),u)\exp\left(\int_{(\zeta^{\varepsilon})^{-1}(u)}^t \frac{\varphi_t}{\varphi}(v,\zeta^{\varepsilon}(v))\,\mathrm{d}v\right)\mathrm{d}u.$$

Now, note that $\varphi((\zeta^{\varepsilon})^{-1}(u),u) = -\int_{(\zeta^{\varepsilon})^{-1}(u)}^{u}(\varphi_t/\varphi)(s,u)\,\mathrm{d}s$, and therefore $g^{\varepsilon}(t)$ can be written in

$$g^{\varepsilon}(t) = -\int_0^{\zeta^{\varepsilon}(t)} \tilde{h}^{\varepsilon}(t,u)\left[1-\exp\left(-\int_u^{(\zeta^{\varepsilon})^{-1}(u)}\left(\frac{\varphi_t}{\varphi}(s,\zeta^{\varepsilon}(u)) - \frac{f_t}{f}(s,u)\right)\mathrm{d}s\right)\right]\mathrm{d}u$$
$$-\int_{\zeta^{\varepsilon}(t)}^{t}\tilde{h}^{\varepsilon}(t,u)\,\mathrm{d}u,$$

where $\tilde{h}^{\varepsilon}(t,u) = \exp\int_u^t(\varphi_t/\varphi)(s,\zeta^{\varepsilon}(s))\,\mathrm{d}s$. This proves that $(t,\varepsilon)\mapsto g^{\varepsilon}(t)$ is continuous in $[0,T]\times[0,1]$ since ψ is continuous in z and ε and z^{ε} is continuous in ε as a $W^{1,\infty}$ function. For $\varepsilon=0$, we have $\zeta^0=\theta$ and

$$g^{0}(t) = -\int_0^{\theta(t)} \tilde{h}^{0}(t,u)\left[1-\exp\left(-\int_u^{\theta^{-1}(u)}\left(\frac{f_t}{f}(s,\theta(u)) - \frac{f_t}{f}(s,u)\right)\mathrm{d}s\right)\right]\mathrm{d}u$$
$$-\int_{\theta(t)}^{t}\tilde{h}^{0}(t,u)\,\mathrm{d}u.$$

The second term on the right-hand side is clearly nonpositive. Moreover, since $\theta^{-1}(u)\geq u$ and $(f_t/f)(t,u)$ is decreasing in u, the first one is also nonpositive. Finally, it is clear that $g^0(t)<0$ whenever $\int_0^t h(u)\,\mathrm{d}u>0$. □

We are now able to prove Theorem 3.1 in the special case where the investment rate function x is positive.

Proposition 3.2. *Let $(x,y)\in\mathscr{A}$ be an admissible trading strategy with $x\geq\delta>0$. Suppose that $y>0$ on a Borel subset A with positive measure. Then there exists some $\varepsilon>0$ such that the pair $(x^{\varepsilon},y^{\varepsilon})$ defined by Eqs.* (3.2) *and* (3.3) *is an admissible trading strategy, provides the same consumption rate function, i.e. $c^{x,y}=c^{x^{\varepsilon},y^{\varepsilon}}$ and satisfies $\theta^{\varepsilon}(T)<\theta(T)$.*

Proof. Let $\tau=\inf\{t\geq 0\colon \int_0^t h(u)\,\mathrm{d}u>0\}$. Then, by definition of $(x^{\varepsilon},y^{\varepsilon})$, we have $\theta^{\varepsilon}(t)=\theta(t)$ for $t\in[0,\tau]$. Next, for $t>\tau$, the last lemma ensures the existence of $\varepsilon_t>0$ and an open neighbourhood V_t of t such that $\theta^{\varepsilon}(u)<\theta(u)$ for all $(u,\varepsilon)\in V_t\times[0,\varepsilon_t]$. Now, note that $\bigcup_t V_t$ is an open cover of the compact set $[\tau,T]$. Then there exists a finite subcover $\bigcup_{i=1,\dots,n}V_{t_i}$ of $[\tau,T]$. Defining $\varepsilon=\min\{\varepsilon_{t_i},\ i=1,\dots,n\}$, we see that for all $t\in[\tau,T]$ we have $\theta^{\varepsilon}(t)<\theta(t)$. This proves that $\theta^{\varepsilon}(t)\leq t$ for all $t\in[0,T]$ and therefore $(x^{\varepsilon},y^{\varepsilon})$ is an admissible trading strategy with $\theta^{\varepsilon}(T)<\theta(T)$. □

In order to use the last result for the proof of the general case, we need a final refinement.

Lemma 3.6. *The parameter* ε *in the last proposition can be fixed to any constant in* $(0,1]$.

Proof. Let $\varepsilon^* = \sup\{\varepsilon \in [0,1): \theta^\varepsilon \leq \theta\}$. Then, since θ^ε is continuous in ε, the pair $(x^{\varepsilon^*}, y^{\varepsilon^*})$ is an admissible strategy. Suppose that $\varepsilon^* < 1$, then the investment function x^{ε^*} is positive and $y^{\varepsilon^*} > 0$ on A. We can therefore apply Proposition 3.2 to $(x^{\varepsilon^*}, y^{\varepsilon^*})$ which leads to a contradiction. □

3.2. Proof of Theorem 3.1

Consider an admissible trading strategy $(x, y) \in \mathscr{A}$ such that $x(t)y(t) > 0$ for all $t \in A$, where A has positive (Lebesgue) measure. Recall that the endowment rate function ω is a positive continuous function on $[0, T]$. Then it admits a lower bound $\delta > 0$. For any positive integer n, define the investment function:

$$x_n(t) = x(t) + \frac{\omega(t)}{nS(t)} 1_{\{x<\delta\}}, \quad 0 \leq t \leq T.$$

It is easily checked that the pair (x_n, y) is an admissible trading strategy. Moreover, as defined, the investment rate function is such that $x_n \geq \delta/(nS(0))$ for each n. Then, we can apply Proposition 3.2 together with Lemma 3.6. Using the notations of the previous paragraph, the pair (x_n^1, y_n^1) is an admissible trading strategy such that $c^{x_n, y} = c^{x_n^1, y_n^1}$. From the definition of x_n, it is clear that

$$x_n^1 \to x^1 \in L_+^1 \quad \text{in the sense of } L^1. \tag{3.12}$$

Moreover, for each n, the delay function $\theta_n^1 = \theta^{x_n^1, y_n^1}$ is a nondecreasing function. We now provide a useful lemma.

Lemma 3.7. *The set of nondecreasing right-continuous functions* θ *such that* $0 \leq \theta(t) \leq t$ *is compact in the sense of the Levy metric.* [2]

Proof. From the Helly selection Theorem (see [2], p. 227) or the Prohorov theorem (see [2], p. 37), the set of nondecreasing right-continuous functions f such that $f(0) = 0$ and $f(1) = 1$ is relatively compact for the Levy metric, since the cumulative distribution functions in this set have compact support. The result follows from the fact that, by obvious normalization, the set of nondecreasing right-continuous functions θ such that $0 \leq \theta(t) \leq t$ is a closed subset of the previous set. □

[2] We recall that the Levy distance between two nondecreasing rightcontinuous functions f and g such that $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ is the infimum of those positive α such that $f(x - \alpha) - \alpha \leq g(x) \leq f(x + \alpha) + \alpha$ for all x.

This proves that

$$\theta_n^1 \to \theta^1 \quad \text{in the sense of the Levy metric,} \tag{3.13}$$

possibly along some subsequence, where θ^1 is a nondecreasing right-continuous function. Since $\theta_n^1 \leq \theta_n$ for all n, we have

$$\theta^1(t) \leq \theta(t) \quad \text{for all } t \in [0, T].$$

We next use a result from Hennequin and Tortrat [6], p. 194 and 198 which says that the Levy convergence implies the pointwise convergence at each continuity point of the limit function. From Eq. (3.3) defining y_n^1 and since any nondecreasing function is continuous a.e., we can then conclude that

$$y_n^1 \to y^1 \in L_+^1 \quad \text{a.e.} \tag{3.14}$$

with

$$y^1(t) = \frac{y(t) f(t, \theta(t)) - h(t)}{f(t, \theta^1(t))} t \in [0, T] \quad \text{a.e.}$$

It remains to prove that θ^1 is the delay function associated to the perturbed trading strategy (x^1, y^1), i.e. $\theta^1 = \theta^{x^1, y^1}$. To see this recall that

$$\int_0^t y_n^1(u)\, \mathrm{d}u = \int_0^{\theta_n^1(t)} x_n^1(u)\, \mathrm{d}u, \quad 0 \leq t \leq T.$$

From Eq. (3.14) together with the dominated convergence theorem, the left-hand-side term converges to $\int_0^t y^1(u)\, \mathrm{d}u$ as $n \to \infty$ (use Eq. (3.3) to bound y_n^1). Next, note that

$$\begin{aligned} \left| \int_0^{\theta_n^1(t)} x_n^1(u)\, \mathrm{d}u - \int_0^{\theta^1(t)} x^1(u)\, \mathrm{d}u \right| &\leq \int_0^{\theta_n^1(t)} |x_n^1(u) - x^1(u)|\, \mathrm{d}u + \left| \int_{\theta_n^1(t)}^{\theta^1(t)} x^1(u)\, \mathrm{d}u \right| \\ &\leq \int_0^T |x_n^1(u) - x^1(u)|\, \mathrm{d}u + \left| \int_{\theta_n^1(t)}^{\theta^1(t)} x^1(u)\, \mathrm{d}u \right|. \end{aligned}$$

From Eqs. (3.12) and (3.13), the right-hand-side term of the last inequality converges to zero as $n \to \infty$. We have then proved that

$$\int_0^t y^1(u)\, \mathrm{d}u \leq \int_0^{\theta^1(t)} x^1(u)\, \mathrm{d}u, \quad 0 \leq t \leq T \text{ a.e.}$$

This shows that $\theta^1 = \theta^{x^1, y^1}$. □

4. First-order condition for a smooth delay function

4.1. Investment regimes

As noted in the previous sections, given an admissible trading strategy (x, y), the associated delay function $\theta^{x,y}$ presents some jumps by definition. This is consistent with the interpretation of θ as the purchasing time of the last asset sold out by the investor.

In the rest of the paper, we derive the first-order conditions for the optimal investment problem with taxes in case where the optimal delay function $\theta^* = \theta^{x^*,y^*}$ is known to be smooth. The study of the general case is left for future work.

Assumption 4.1. The utility maximization problem (2.8) admits a solution $(x^*, y^*) \in \mathscr{A}$ such that $\theta^* = \theta^{x^*,y^*}$ is continuous on $[0, T]$.

Assuming continuity of the delay function θ^* is a strong assumption and imposes, in particular, some restrictions on the associated optimal investment function x^*.

Proposition 4.1. *Let (x, y) be a nonzero admissible trading strategy such that $\theta^{x,y}$ is continuous on $[0, T]$ and*

$$\int_0^T x(u)\,\mathrm{d}u = \int_0^T y(u)\,\mathrm{d}u. \tag{4.1}$$

Then there exist $0 \le b^x < c^x < T$ such that:

(i) *No market participation regime: on $[0, b^x]$, $x = y = 0$ a.e. and therefore $\theta^{x,y}(t) = t$,*

(ii) *Investment regime: on $[b^x, c^x]$, $x > 0$ and $y = 0$ a.e. and therefore $\theta(t) = b$,*

(iii) *No-investment regime: on $[c^x, T]$, $x = 0$ a.e, $\theta^{x,y}(T) = c$ and there exists an increasing sequence $(t_n)_n$ valued in (c, T) such that $\theta^{x,y}(t_n) < c$ and $\theta^{x,y}(t_n) \to c$.*

Proof. Since $x \neq 0$, we can define

$$b^x = \inf\left\{t \in [0, T]\colon \int_0^t x(u)\,\mathrm{d}u > 0\right\}, \tag{4.2}$$

$$c^x = \inf\left\{t \in [b, T]\colon \int_t^T x(u)\,\mathrm{d}u = 0\right\}. \tag{4.3}$$

It is clear that we have $0 \le b < c \le T$. From Lemma 2.1(iii), it follows that we must have $x > 0$ on $[b^x, c^x]$ a.e. in order for $\theta^{x,y}$ to be continuous. The rest of the claim follows easily from condition (4.1), the continuity of $\theta^{x,y}$ as well as Theorem 3.1. □

Note that condition (4.1) is clearly satisfied by the optimal trading strategy (x^*, y^*); this is an immediate consequence of the increasing feature of the utility function U in the c variable.

The next step is to rewrite the optimal control problem (2.8) in terms of $(\hat{x}, \theta)$ rather than (x, y) where $\hat{x} = xS$ is the time t investment in the financial asset in units of the consumption good. This is an easy consequence of Lemma 2.1(ii) which says that $\int_0^t y(u)\,\mathrm{d}u = \int_0^{\theta(t)} x(u)\,\mathrm{d}u$. Since θ is a nondecreasing function (then differentiable a.e. on $[0, T]$ by the Lebesgue theorem) and $(x, y) \in L^1_+ \times L^1_+$, this provides $y(t) = \dot{\theta}(t)x(\theta(t))$ a.e. on $[0, T]$ where the dot denotes the derivative with repect to the t variable. Therefore, the consumption rate function (2.5) can be written in terms of $(\hat{x}, \theta) = (xS, \theta)$ as

$$c^{\hat{x},\theta}(t) = \omega(t) - \hat{x}(t) + \dot{\theta}(t)\hat{x}(\theta(t))\varphi(t, \theta(t)), \quad t \in [0, T] \text{ a.e.} \tag{4.4}$$

In order to derive the first-order conditions by means of the variations calculus, we have to assume further regularity conditions on the optimal trading strategy of Assumption 4.1.

Assumption 4.2. The optimal trading strategy (x^*, y^*) of Assumption 4.1 is such that
(i) the investment rate function x^* is piecewise C^1 on $[0, T]$,
(ii) the delay function $\theta^* = \theta^{x^*,y^*}$ is piecewise C^1 on $[0, T]$.

Remark 4.1. *Since the financial asset price function $S(t)$ is C^1 and positive on $[0, T]$, Assumption 4.2(i) says equivalently that $\hat{x}^*$ is piecewise C^1 on $[0, T]$.*

Remark 4.2. *Let x be an investment function with $0 \le x \le \omega$ and consider a nondecreasing function θ with $0 \le \theta(t) \le t$ for all $t \in [0, T]$. Suppose that θ is continuous and piecewise C^1 (as required in Assumptions 4.1 and 4.2(ii) for the optimal strategy). Then, the associated disinvestment function $y = \dot{\theta}x(\theta)$ is such that $(x, y) \in \mathscr{A}$. In the sequel we shall identify the pairs $(\hat{x}, \theta)$ and (x, y). In particular $(\hat{x}, \theta)$ will be referred to as a trading strategy.*

We end this paragraph by discussing part(ii) of the last assumption. Define the function X^* on $[b^{x^*}, c^{x^*}]$ by $X^*(t) = \int_0^t x^*(u)\,\mathrm{d}u$. As a continuous increasing function, X^* admits an inverse function X^{*-1} defined on $X^*([b^{x^*}, c^{x^*}]) = [0, \int_0^T x^*(u)\,\mathrm{d}u]$. Next, let Y^* be the function defined on $[0, T]$ by $Y^*(t) = \int_0^t y^*(u)\,\mathrm{d}u$. Then, from Lemma 2.1(ii) and the no short sales condition (2.3), we have $\theta^*(t) = X^{*-1}(Y^*(t))$. Since $x^* > 0$ on $[b^{x^*}, c^{x^*}]$ a.e., this provides

$$\dot{\theta}^*(t) = \frac{y^*(t)}{x^*(\theta^*(t))}, \quad t \in [b^{x^*}, T] \text{ a.e.}$$

Therefore, Assumption 4.2(ii) imposes that the function $t \mapsto y^*(t)/x^*(\theta^*(t))$ be defined and piecewise continuous on $[b^{x^*}, T]$.

4.2. Necessary conditions from the calculus of variations

Under Assumptions 4.1 and 4.2, the utility maximization problem (2.8) can be written in

$$\sup \int_0^T U(t, c(t))\,\mathrm{d}t; \quad c(t) = \omega(t) - \hat{x}(t) + v(t)\hat{x}(\theta(t))\varphi(t, \theta(t)), \tag{4.5}$$

where $\hat{x}$ is piecewise C^1 and v is piecewise continuous on $[0,T]$. $(\hat{x},v)$ are controls subject to the constraints:

$$0 \le \hat{x}(t) \le \omega(t), \quad \forall t \in [0,T], \tag{4.6}$$

$$v(t) \ge 0, \quad \forall t \in [0,T]. \tag{4.7}$$

The function θ is a continous state variable defined by the dynamics

$$\dot{\theta}(t) = v(t) \quad \text{and} \quad \theta(0) = 0 \tag{4.8}$$

with state constraints

$$0 \le \theta(t) \le t \quad \text{for any } t \in [0,T]. \tag{4.9}$$

The originality of this optimal control problem consists in the presence of the state variable θ as argument of the control variable $\hat{x}$, which makes the classical first order conditions of optimality not valid in this context. In this section, we shall provide necessary conditions of optimality for this problem by adapting the classical variational approach which allows to derive the Pontryagin principle.

Throughout this paragraph, v and $\hat{x}$ denote optimal control variables of the problem and θ is the induced state variable. In order to derive necessary conditions for optimality of v and $\hat{x}$, we consider small variations in the form:

$$v(t,\varepsilon) = v(t) + \varepsilon\delta^v(t) \quad \text{and} \quad \hat{x}(t,\varepsilon) = \hat{x}(t) + \varepsilon\delta^{\hat{x}}(t)$$

where $\varepsilon \ge 0$, δ^v is a piecewise continuous function on $[0,T]$ and $\delta^{\hat{x}}$ is differentiable on $[0,T]$ with bounded derivative. We also denote by $\theta(t,\varepsilon)$ the solution of Eq. (4.8) where $v(t)$ is replaced by $v(t,\varepsilon)$ and we define

$$c(t,\varepsilon) = \omega(t) - \hat{x}(t,\varepsilon) + v(t,\varepsilon)\hat{x}(\theta(t,\varepsilon),\varepsilon)\varphi(t,\theta(t,\varepsilon)),$$

we shall keep using the notation $\dot{\theta}(t,\varepsilon)$ for the derivative with respect to the t variable. Now, consider the function:

$$J(\hat{x} + \varepsilon\delta^{\hat{x}}, v + \varepsilon\delta^v) = \int_0^T U(t, c(t,\varepsilon))\,\mathrm{d}t. \tag{4.10}$$

For sake of simplicity, we shall denote $\hat{x}(t,0) = \hat{x}(t)$. Since $(\hat{x},v)$ is assumed to be an optimal control, we have by definition that $J(\hat{x},v)$ is the value function of the optimal control problem of interest.

Remark 4.3. *Suppose that $\hat{x}(t,a)$, $v(t,a)$ and $\theta(t,a)$ satisfy the constraints* (4.6), (4.7) *and* (4.9) *for some $a > 0$, i.e. $(\hat{x}(.,a), v(.,a))$ is an admissible control. Then, for any $\varepsilon \in [0,a]$, $(\hat{x}(.,\varepsilon), v(.,\varepsilon))$ is an admissible control and $(\hat{x},v)$ is said to be a radial point of the set of admissible controls in the direction $(\delta^{\hat{x}},\delta^v)$. For such a pair $(\delta^{\hat{x}},\delta^v)$, the necessary conditions for a maximum provide $J(\hat{x},v) \ge J(\hat{x}+\varepsilon\delta^{\hat{x}}, v+\varepsilon\delta^v)$. Whenever the right hand derivative with respect to ε of $J(\hat{x}+\varepsilon\delta^{\hat{x}}, v+\varepsilon\delta^v)$ exists, we* *must have*

$$\frac{\mathrm{d}^+}{\mathrm{d}\varepsilon} J(\hat{x}+\varepsilon\delta^{\hat{x}}, v+\varepsilon\delta^{v})\bigg|_{\varepsilon=0} \le 0,$$

where d^+ denotes the right derivative.

Remark 4.4. *If $(\hat{x}, v)$ is a radial point (in the set of all admissible controls) both in the direction $(\delta^{\hat{x}}, \delta^{v})$ and $(-\delta^{\hat{x}}, -\delta^{v})$ then $(\hat{x}, v)$ is said to be an internal point in the direction $(\delta^{\hat{x}}, \delta^{v})$. For such pair $(\delta^{\hat{x}}, \delta^{v})$, whenever the derivative with respect to ε of $J(\hat{x}+\varepsilon\delta^{\hat{x}}, v+\varepsilon\delta^{v})$ exists, we must have*

$$\delta J(\hat{x}, v, \delta^{\hat{x}}, \delta^{v}) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} J(\hat{x}+\varepsilon\delta^{\hat{x}}, v+\varepsilon\delta^{v})\bigg|_{\varepsilon=0} = 0.$$

The last remarks are the basic tools in order to derive the Pontryagin principle. By adapting the classical methods to our optimal control problem with endogeneous delay, we obtain the following result.

Proposition 4.2. *Suppose that the optimal control $(\hat{x}, v)$ is a radial point of the set of admissible controls in the direction $(\delta^{\hat{x}}, \delta^{v})$. Then, we have*

$$\int_0^T \delta^{v}(t)\psi(t)\,\mathrm{d}t + \int_0^T \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t \le 0,$$

where

$$\psi(t) = -\int_t^T [\dot{\hat{x}}(\theta(s))\varphi(s,\theta(s)) + \hat{x}(\theta(s))\varphi_\theta(s,\theta(s))]U_c(s,c(s))v(s)\,\mathrm{d}s + \hat{x}(\theta(t))\varphi(t,\theta(t))U_c(t,c(t)), \tag{4.11}$$

$$\phi(t) = -U_c(t,c(t)) + \varphi(\theta^{-1}(t),t)U_c(\theta^{-1}(t),c(\theta^{-1}(t)))1_{\{t\le\theta(T)\}} \tag{4.12}$$

for all $t \in [0,T]$.

Proof. See the appendix. □

Remark 4.5. *From the definition of the utility function $U(t,c)$ as well as Assumptions* 4.1 *and* 4.2, *it is easily checked that ψ is continuous and piecewise C^1 on the interval* $[0,T]$.

We now derive necessary conditions of optimality by considering variations $(\delta^{\hat{x}}, \delta^{v})$ such that the optimal control $(\hat{x}, v)$ is a radial point in the direction $(\delta^{\hat{x}}, \delta^{v})$. We shall consider separately directions $(0, \delta^{v})$ and $(\delta^{\hat{x}}, 0)$ and we will show that this suffices to recover all first-order conditions.

Lemma 4.1. *Let $t^* \in (0,T]$ such that θ is increasing on a neighbourhood V of t^* in $[0,T]$ and $0<\theta(t^*)<t^*$. Then $\dot{\psi}=0$ on V.*

Proof. Since θ is increasing on V, the set $W=\{t\in V:\ \dot\theta(t)>0\}$ is dense in V. Let t be any element of W and let $[t_1,t_2]$ be a neighbourhood of t such that $\dot\theta(s)\geq\alpha$ for some $\alpha>0$ and $0<\theta(s)<s$ for any $s\in[t_1,t_2]$. Let $k=\min_{t_1\leq s\leq t_2}\{s-\theta(s),\ \theta(s)\}$ which is obviously positive. Consider some C^1 function h with support in $[t_1,t_2]$ and choose $\varepsilon>0$ such that $\varepsilon\|h\|_\infty\leq k$ and $\varepsilon\|\dot h\|_\infty\leq\alpha$ which proves that $\theta+\varepsilon h$ is nondecreasing and $0\leq\theta(t)+\varepsilon h(t)\leq t$ for $0\leq t\leq T$. Then both $v+\varepsilon\dot h$ and $v-\varepsilon\dot h$ are admissible controls. Therefore, letting $\delta^v=\dot h$ and $\delta^{\hat x}=0$, we see that $(\hat x,v)$ is an internal point in the direction $(\delta^{\hat x},\delta^v)$ and the first-order condition for the optimality of $(\hat x,v)$ is given by $\delta J(\hat x,v;\delta^{\hat x},\delta^v)=0$. This provides

$$\int_0^T \dot h(s)\psi(s)\,\mathrm{d}s=\int_{t_1}^{t_2}\dot h(s)\psi(s)\,\mathrm{d}s=0.$$

Integrating by parts, we get $\int_{t_1}^{t_2}h(s)\dot\psi(s)\,\mathrm{d}s=0$ for any C^1 function h with support in $[t_1,t_2]$, which proves that $\dot\psi=0$ on $[t_1,t_2]$. The result of the lemma follows from the density of W in V.

Lemma 4.2. *Let $[t_0,t_1]$ be an interval on which $\theta(t)=t$. Then we have $\dot\psi(t)\leq 0$ for all $t\in[t_0,t_1]$.*

Proof. Let h be some nonpositive C^1 function with support in $[t_0,t_1]$. Then, possibly multiplying by a constant, we can assume that $\|\dot h\|_\infty\leq 1$ and therefore $v+\dot h$ is an admissible control and $(\hat x,v)$ is a radial point in the direction $(0,\dot h)$. This proves that $\int_0^T\psi(t)\dot h(t)\,\mathrm{d}t=\int_{t_0}^{t_1}\psi(t)\dot h(t)\,\mathrm{d}t=-\int_{t_0}^{t_1}\dot\psi(t)h(t)\,\mathrm{d}t\leq 0$. The result follows from the arbitrariness of the nonpositive C^1 function h. □

Lemma 4.3. *Let $[t_1,t_2]\subset[0,T)$ be a maximal interval on which $\dot\theta=0$. Then we have $\psi(t_1)=\psi(t_2)$ and $\psi(t)\leq\psi(t_1)$ for all $t\in[t_1,t_2]$.*

Proof. The proof is organized in two steps. We first prove that $\psi(t)\leq\psi(t_2)$ for all $t\in[t_1,t_2]$ and then $\psi(t)\leq\psi(t_1)$ for all $t\in[t_1,t_2]$.

(i) Consider some $\eta>0$. Since $[t_1,t_2]$ is a maximal interval on which $\dot\theta=0$, there exist t_3 and t_4 in the interval $(t_2,t_2+\eta]$ such that $\dot\theta(t)>0$ on $[t_3,t_4]$; we then introduce $\Delta=\theta(t_4)-\theta(t_3)$.

Let h be any (nonzero) nondecreasing C^1 function on $[t_1,t_2]$ with $h(t_1)=0$. Possibly multiplying by some well chosen positive constant, we can assume $h(t_2)=\Delta$. We next extend h continuously to $[0,T]$ as follows:

- $\dot h=0$ on $[t_2,t_3]$,
- $\dot h=-\dot\theta$ on $[t_3,t_4]$,
- $h=0$ outside the interval $[t_1,t_4]$.

Then, it is easily checked that $v+\dot h$ is an admissible control and therefore $(\hat x,v)$ is a radial point in the direction $(0,\dot h)$. This proves that $\int_0^T\psi(t)\dot h(t)\,\mathrm{d}t=\int_{t_1}^{t_2}\psi(t)\dot h(t)\,\mathrm{d}t+\int_{t_3}^{t_4}\psi(t)\dot h(t)\,\mathrm{d}t\leq 0$. Integrating by parts and noting that $\dot\psi=0$ on $[t_3,t_4]$ by Lemma 4.1, this provides: $\int_{t_1}^{t_2}\psi(t)\dot h(t)\,\mathrm{d}t-\psi(t_3)h(t_3)\leq 0$. Now, since $h(t_3)=h(t_2)$ and $h(t_1)=0$, we obtain $\int_{t_1}^{t_2}[\psi(t) - \psi(t_3)]\dot{h}(t)\,dt \le 0$ for all nonnegative function $\dot{h}$ on $[t_1, t_2]$ which implies that $\psi(t) \le \psi(t_3)$ for all $t \in [t_1, t_2]$. By sending η to zero and using the continuity of ψ, we get $\psi(t) \le \psi(t_2)$ for all $t \in [t_1, t_2]$.

(ii) We now prove that $\psi(t) \le \psi(t_1)$ for all $t \in [t_1, t_2]$. Consider some $\eta > 0$. Since $[t_1, t_2]$ is a maximal interval on which $\dot{\theta} = 0$, there exist t_{-1} and t_0 in the interval $[t_1 - \eta, t_1)$ such that $\dot{\theta} > 0$ on $[t_{-1}, t_0]$. Let t^* be an arbitrary value in $[t_1, t_2]$. Given a sufficiently small $\varepsilon > 0$, consider the following continuous function h defined on $[0, T]$ by

- $h = 0$ outside the interval $[t_{-1}, t^* + \varepsilon]$,
- $\dot{h} = -\dot{\theta}$ on $[t_{-1}, t_0]$,
- $\dot{h} = 0$ on $[t_0, t^* - \varepsilon]$,
- $\dot{h} = [\theta(t_0) - \theta(t_{-1})]/2\varepsilon$ on $[t^* - \varepsilon, t^* + \varepsilon]$.

Then it is easily checked that $v + \dot{h}$ is an admissible control and therefore $(\hat{x}, v)$ is a radial point in the direction $(0, \dot{h})$. It then follows by the necessary conditions of optimality of $(\hat{x}, v)$ that $\int_0^T \psi(t)\dot{h}(t)\,dt \le 0$. Letting ε go to zero, this provides

$$\psi(t^*) \le \frac{\int_{t_{-1}}^{t_0} \psi(t)\dot{\theta}(t)\,dt}{\theta(t_0) - \theta(t_{-1})}$$

by the continuity of ψ. The required result is obtained by sending η to zero in the last inequality and using again the continuity of ψ. □

Lemma 4.4. *Let $T_0 \in (0, T]$ such that $\theta(t) = c < T_0$ for all $t \in [T_0, T]$. Suppose that there exists an increasing sequence $(t_n)_n$ converging to T_0 such that $0 < \theta(t_n) < c$. Then $\psi(T_0) = 0$.*

Proof. For each $n \in N$, define the function:

$$h^n(t) = \begin{cases} \theta(t) - \theta(t_n), & t_n \le t \le T, \\ 0, & 0 \le t \le t_n \end{cases}$$

and define the disjoint subsets of $[0, T]$

$$C_n = \{t \in [t_n, T_0]\colon \dot{\theta}(t) = 0\} \quad \text{and} \quad D_n = [t_n, T] \setminus C_n.$$

Then $\text{int}(C_n) = \cup_{t \in \text{int}(C_n)} (a_t, b_t)$ where $[a_t, b_t]$ is a maximal interval of C_n containing t. As defined, the intervals (a_t, b_t) and (a_s, b_s), $t, s \in \text{int}(C_n)$ are either identical or disjoint. Therefore, we can write $\text{int}(C_n) = \cup_{i \in I} (a_i, b_i)$ for a possibly infinite family I. Note also that $b_i \neq T_0$ for all $i \in I$ since there exists an increasing sequence $(t_n)_n$ converging to T_0 such that $\theta(t_n) < \theta(T_0)$.

Now, for each $i \in I$, we have $\psi(a_i) = \psi(b_i)$ by Lemma 4.3 and therefore,

$$\int_{C_n} \dot{\psi} = \sum_{i \in I} \int_{a_i}^{b_i} \dot{\psi} = 0. \tag{4.13}$$

Moreover, since h^n is constant on each $[a_i, b_i]$, integration by parts provides

$$\int_{C_n} \dot{\psi} h^n = \sum_{i \in I} \int_{a_i}^{b_i} \dot{\psi} h^n = 0. \tag{4.14}$$

Now, it is easily checked that $v - \varepsilon \dot{h}^n$ is an admissible control for small $\varepsilon > 0$ and therefore $(\hat{x}, q)$ is a radial point in the direction $(0, -\dot{h}^n)$. This provides the optimality condition

$$\int_0^T \dot{h}^n(t)\psi(t)\,\mathrm{d}t = \int_{t_n}^{T_0} \dot{h}^n(t)\psi(t)\,\mathrm{d}t \geq 0, \tag{4.15}$$

where we used the fact that h^n is constant on $[T_0, T]$. Integrating by parts and recalling that $h^n(t_n) = 0$, we get

$$\psi(T_0)h^n(T_0) - \int_{D_n} \dot{\psi}(t)h^n(t)\,\mathrm{d}t \geq 0,$$

where we used Eq. (4.14). This provides

$$\begin{aligned}\psi(T) &\geq \int_{D_n} \dot{\psi}(t)\frac{h^n(t)}{h^n(T)}\,\mathrm{d}t \\ &\geq \int_{D_n} \dot{\psi}(t)\,\mathrm{d}t,\end{aligned}$$

where the second inequality follows from the fact that $h^n(t) \leq h^n(T)$ and $\dot{\psi} \leq 0$ on D_n by Lemmas 4.1 and 4.2. Using Eq. (4.13), we then get

$$\psi(T) \geq \int_{t_n}^T \dot{\psi}(t)\,\mathrm{d}t = \psi(T) - \psi(t_n),$$

which provides $\psi(T) \geq 0$ by sending n to ∞ and using the continuity of ψ.

To see that the reverse inequality holds, notice that, since $\theta(T_0) < T_0$, it follows that $v + \varepsilon \dot{h}^n$ is also an admissible control for small $\varepsilon > 0$. Therefore, $(\hat{x}, v)$ is a radial point in the direction $(0, \dot{h}^n)$. Repeating the above arguments provides the required result. □

In the following lemmas of this section, we concentrate on the control variable $\hat{x}$ and we consider variations of the control variable $(\hat{x}, v)$ of the form $(\delta^{\hat{x}}, 0)$.

Lemma 4.5. *Let $t^* \in [0, \theta(T)]$ such that $0 < \hat{x}(t^*) < \omega(t^*)$. Then $\phi(t^*) = 0$.*

Proof. Since $\hat{x}$ is continuous there exits an interval $[t_1, t_2] \subset [0, \theta(T)]$ containing t^* such that $0 < \hat{x} < \omega$ on $[t_1, t_2]$. Let $\delta^{\hat{x}}$ be any continuous function with compact support on $[t_1, t_2]$. Then, possibly multiplying by a constant both $\hat{x} + \delta^{\hat{x}}$ and $\hat{x} - \delta^{\hat{x}}$ are admissible controls and therefore $(\hat{x}, v)$ is an internal point in the direction $(\delta^{\hat{x}}, 0)$. This provides the optimality condition:

$$0 = \int_0^T \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t.$$

The required result follows from the arbitrariness of the test function $\delta^{\hat{x}}$. □

Lemma 4.6. (i) *Let* $[t_1, t_2]$ *be an interval of* $[0, T]$ *on which* $\hat{x} = 0$. *Then* $\phi(t) \le 0$.

Proof. Let $\delta^{\hat{x}}$ be any nonnegative continuous function with compact support on $[t_1, t_2]$. Then $\delta^{\hat{x}}$ is an admissible control and therefore $(\hat{x}, v)$ is a radial point in the direction $(\delta^{\hat{x}}, 0)$. This provides the optimality condition $\int_{t_1}^{t_2} \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t \le 0$. The required result follows from the arbitrariness of the test function $\delta^{\hat{x}}$. □

Lemma 4.7. *Let* $[t_1, t_2] \subset [0, T]$ *be an interval on which* $\hat{x} = \omega$. *Then* $\phi(t) \ge 0$ *for any* t *in* $[t_1, t_2]$.

Proof. Let $\delta^{\hat{x}}$ be any negative continuous function with compact support on $[t_1, t_2]$. Then, possibly multiplying by a constant $\hat{x} + \delta^{\hat{x}}$ is an admissible control and therefore $(\hat{x}, v)$ is a radial point in the direction $(\delta^{\hat{x}}, 0)$. This provides the optimality condition:

$$0 \ge \int_0^T \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t = \int_{t_1}^{t_2} \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t.$$

The required result follows from the arbitrariness of the test function $\delta^{\hat{x}}$. □

We now use the results of the previous lemmas in order to derive the first-order conditions associated with the optimal investment problem with taxes in the case where the optimal investment function and the optimal delay function are known to satisfy Assumptions 4.1 and 4.2.

Theorem 4.1. *Consider an admissible strategy* $(\hat{x}, \theta) \in \mathscr{A}_0$ *and let* b^x *and* c^x *be the associated dates of regime changes defined in Proposition* 4.1. *Then, the following conditions of optimality hold*:

(i) $\dot{\psi}(t) \le 0$ *for all* $t \in [0, b^x]$,

(ii) $\psi(b^x) = \psi(c^x)$ *and* $\psi(t) \le \psi(b^x)$ *for all* $t \in [b^x, c^x]$,

(iii) *for all* $t \in [c^x, T]$, $\dot{\psi}(t) = 0$ *if* $\dot{\theta}(t) > 0$. *If* $\dot{\theta} = 0$ *on some* $(t_1, t_2) \subset [c^x, T]$ *then* $\psi(t_1) = \psi(t_2)$ *and* $\psi(t) \le \psi(t_1)$ *for all* $t \in [t_1, t_2]$,

(iv) *Let* $T_0 \in (0, T]$ *such that* $[T_0, T]$ *is a maximal interval on which* θ *is constant. Then* $\psi(T_0) = 0$ *and* $\psi(t) \le \psi(T_0)$ *for all* $t \in [T_0, T]$.

(v) *for all* $t \in [b^x, c^x]$, $\phi(t) = 0$ *if* $\hat{x}(t) < \omega(t)$ *and* $\phi(t) \ge 0$ *if* $\hat{x}(t) = \omega(t)$.

Conversely, suppose that $(\hat{x}, \theta) \in \mathscr{A}_0$ *satisfy the necessary conditions* (i)–(v). *Then for any variation* $(\delta^{\hat{x}}, \delta^v)$ *such that* $(\hat{x}, v)$ *is a radial point in the direction* $(\delta^{\hat{x}}, \delta^v)$, *we have*

$$\delta J(\hat{x}, v; \delta^{\hat{x}}, \delta^v) \le 0,$$

where $\delta J(\hat{x}, v; \delta^{\hat{x}}, \delta^v)$ is the right-side derivative of the value function J defined in Eq. (4.10) *in the direction* $(\delta^{\hat{x}}, \delta^v)$.

Proof. The necessary conditions (i)–(v) are obtained by direct application of Lemmas 4.1–4.7. We now prove the second part of the theorem. Note that $\hat{x} = 0$ on $[c^x, T]$ and therefore only nonnegative perturbations $\delta^{\hat{x}}$ are admissible. Since $U_c(t, c(t)) \geq 0$, we have

$$\delta J(\hat{x}, v; \delta^{\hat{x}}, \delta^v) \leq \int_0^T \delta^v(t)\psi(t)\,\mathrm{d}t + \int_0^{c^x} \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t.$$

For $t \in [0, b^x]$, $\theta^{-1}(t) = t$ and therefore $\phi(t) = 0$. Next, for $t \in \{t \in [b^x, c^x]: \hat{x}(t) < \omega(t)\}$, we have $\phi(t) = 0$ by (v). Now, note that for $t \in \{s \in [b^x, c^x]: \hat{x}(s) = \omega(s)\}$, $\phi(t) \geq 0$, by (v), and $\delta^{\hat{x}}$ must be nonpositive in order for the variation $\delta^{\hat{x}}$ to be admissible. This proves that $\int_0^{c^x} \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t \leq 0$ and therefore

$$\delta J(\hat{x}, v; \delta^{\hat{x}}, \delta^v) \leq \int_0^{b^x} \delta^v\psi + \int_{b^x}^{c^x} \delta^v\psi + \int_{c^x}^{T_0} \delta^v\psi + \int_{T_0}^{T} \delta^v\psi, \tag{4.16}$$

where T_0 is as defined in the theorem. By condition (iv), we have $\psi(t) \leq \psi(T_0)$ for all $t \in [T_0, T]$ and the variation δ^v must be nonnegative therein in order to be admissible. This proves that

$$\int_{T_0}^{T} \delta^v(t)\psi(t)\,\mathrm{d}t \leq 0. \tag{4.17}$$

Next, define $h(t) = \int_0^t \delta^v(s)\,\mathrm{d}s$, $0 \leq t \leq T$. Then, integrating by parts in the first integral on the right-hand side of Eq. (4.16), we get

$$\int_0^{b^x} \delta^v(t)\psi(t)\,\mathrm{d}t = h(b^x)\psi(b^x) - \int_0^{b^x} h(t)\dot{\psi}(t)\,\mathrm{d}t.$$

Now, $\dot{\psi} \leq 0$ on $[0, b^x]$ by (i) and, since $\theta(t) = t$ on $[0, b^x]$, h must be nonpositive therein, in order for δ^v to be an admissible variation, and we have

$$\int_0^{b^x} \delta^v(t)\psi(t)\,\mathrm{d}t \leq h(b^x)\psi(b^x). \tag{4.18}$$

Since $\dot{\theta} = 0$ on $[b^x, c^x]$, δ^v must be nonnegative therein in order to be an admissible variation. Since $\psi(t) \leq \psi(b^x) = \psi(c^x)$ by (ii), this provides

$$\int_{b^x}^{c^x} \delta^v\psi \leq \psi(b^x)[h(c^x) - h(b^x)] = \psi(c^x)h(c^x) - \psi(b^x)h(b^x). \tag{4.19}$$

Finally, let $\cup_{i \in I}[a_i, b_i]$ be the closure of $\{t \in [c, T_0]: \dot{\theta}(t) = 0\}$ where the intervals $[a_i, b_i]$ are disjoint (see the proof of Lemma 4.4 for the construction of such intervals). Then, integrating by parts and using (iii) and (iv), we get

$$\int_{c^x}^{T_0} \delta^v(t)\psi(t)\,\mathrm{d}t \le -h(c^x)\psi(c^x) - \sum_{i\in I}\left\{\psi(a_i)[h(b_i)-h(a_i)] - \int_{a_i}^{b_i} \delta^v(t)\psi(t)\,\mathrm{d}t\right\}.$$

Now, δ^v must be nonnegative on each interval $[a_i, b_i]$ in order to be an admissible variation. Since $\psi(t)\le\psi(a_i)$ for $t\in[a_i,b_i]$ by (iii), this shows that each term inside the sum is zero and therefore

$$\int_{c^x}^{T} \delta^v(t)\psi(t)\,\mathrm{d}t \le -h(c^x)\psi(c^x). \tag{4.20}$$

The required result is obtained by plugging Eqs. (4.17)–(4.20) into (4.16). □

Appendix. Proof of Proposition 4.2

We denote by $\theta(t,\varepsilon)$ the solution of Eq. (4.8) where $v(t)$ is replaced by $v(t,\varepsilon)$; we shall keep using the notation $\dot\theta(t,\varepsilon)$ for the derivative with respect to the variable t.

Remark that $\theta(t,\varepsilon)$ is defined as the unique solution to $\dot\theta(t,\varepsilon) = v(t) + \varepsilon\delta^v(t)$, with initial condition $\theta(0,\varepsilon)=0$, and is therefore continuously differentiable with respect to ε.

Now, let λ be some a.e. differentiable function defined on $[0,T]$ with $\lambda(T)=0$ and consider:

$$\begin{aligned} J(\hat x + \varepsilon\delta^{\hat x}, v+\varepsilon\delta^v) &= \int_0^T U(t, c(t,\varepsilon))\,\mathrm{d}t \\ &= \int_0^T \{U(t,c(t,\varepsilon)) - \lambda(t)[\dot\theta(t,\varepsilon) - v(t,\varepsilon)]\}\,\mathrm{d}t, \end{aligned}$$

where $c(t,\varepsilon) = \omega(t) - (\hat x + \varepsilon\delta^{\hat x}) + \dot\theta(t,\varepsilon)(\hat x + \varepsilon\delta^{\hat x})(\theta(t,\varepsilon))\phi(t,\theta(t,\varepsilon))$.

Integrating by parts in Eq. (4.10) and recalling that $\lambda(T)=\theta(0,\varepsilon)=0$, we get

$$J(\hat x + \varepsilon\delta^{\hat x}, v+\varepsilon\delta^v) = \int_0^T [U(t,c(t,\varepsilon)) + \lambda(t)v(t,\varepsilon) + \dot\lambda(t)\theta(t,\varepsilon)]\,\mathrm{d}t.$$

The derivative of J with respect to ε, evaluated at $\varepsilon=0$, is then given by

$$\begin{aligned} &\delta J(\hat x, v; \delta^{\hat x}, \delta^v) \\ &\quad= \int_0^T \{\theta_\varepsilon(t,0)[\dot\lambda(t) + U_c(t,c(t))v(t)[\hat x(\theta(t))\varphi_\theta(t,\theta(t)) + \dot{\hat x}(\theta(t))\varphi(t,\theta(t))]] \\ &\qquad + \delta^v(t)[U_c(t,c(t,0))\hat x\circ\theta(t)\varphi(t,\theta) + \lambda(t)] \\ &\qquad - \delta^{\hat x}(t)U_c(t,c(t,0)) + \delta^{\hat x}(\theta(t))U_c(t,c(t,0))v(t)\varphi(t,\theta)\}\,\mathrm{d}t, \end{aligned}$$

where we denoted by U_c the partial derivative of U with respect to the consumption variable and θ_ε the derivative of $\theta(t,\varepsilon)$ with respect to ε. The differentiation under the integral sign is justified by Lemma 3.1, p. 6 in Fleming and Rishel [5].

Now, in order to get rid of the first term inside the integral defining $\delta J(\hat{x}, v; \delta^{\hat{x}}, \delta^{v})$, define $\lambda(.)$ by

$$\lambda(t) = -\int_t^T U_c(s, c(s))v(s)[\hat{x}(\theta(s))\varphi_\theta(s, \theta(s)) + \dot{\hat{x}}(\theta(s))\varphi(s, \theta(s))]\,\mathrm{d}s. \tag{A.1}$$

Since $\dot{\hat{x}}$ is bounded and θ is continuous, the function $\lambda(.)$ defined in Eq. (A.1) is continuous and differentiable a.e.

Next, note that, since θ is nondecreasing and piecewise continuously differentiable, it follows from Saard's lemma that the set $\{\theta(s)\colon \dot{\theta}(s) = 0\}$ has zero Lebesgue measure and the inverse function θ^{-1} is defined a.e. Therefore, by the state Eq. (4.8), a direct change of variable provides

$$\delta J(\hat{x}, v; \delta^{\hat{x}}, \delta^{v}) = \int_0^T \delta^{v}(t)\psi(t)\,\mathrm{d}t + \int_0^T \delta^{\hat{x}}(t)\phi(t)\,\mathrm{d}t, \tag{A.2}$$

where $\psi(t)$ and $\phi(t)$ are the functions given in the proposition.

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