Generalized Lipschitz functions

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1. Introduction

The aim of this paper is to establish a compactness result on some function sets. More precisely, our paper extends the concept of Lipschitz functions to a larger class including nondecreasing (nonnecessarily continuous) functions, functions with bounded below derivatives. We prove then that a bounded subset of this set of "generalized" Lipschitz functions is relatively compact. This result as Ascoli's theorem can be applied, in particular, in order to establish existence results for some families of differential equations.

The main idea is very simple: it suffices to change the axis in order to transform a family of nondecreasing functions in Lipschitz ones and then to apply Ascoli's theorem. As we will see, this simple geometrical approach can be extended to a wider class of functions.

The paper is organized as follows. In the next section we shall define the concept of Q-Lipschitz functions, where Q is a convex cone and we shall construct a particular topology on this set. In Section 2, we shall establish our compactness result and we shall explore some properties of the considered topology. In Section 3, we shall extend the previous result to a more general class of functions and in Section 4 we shall present some applications of our result.

At the end of this introduction we recall some useful definitions and notations.

Let Y be a closed subset of R^ℓ then, for every $y \in Y$, the tangent cone $T_Y(y)$ in the sense of Clarke consists of all vectors $v \in R^\ell$ such that, for all sequences $\{t^k\} \subset (0, \infty)$

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and $\{y^k\} \subset Y$ converging, respectively, to 0 and y, there exists a sequence $\{v^k\} \subset R^\ell$ converging to v, with $y^k + t^k v^k \in Y$, for all k. Clarke's normal cone is then defined by polarity as follows:

$$N_Y(y) = T_Y(y)^\circ = \{p \in R^\ell:\ pv \le 0,\ \text{for all } v \in T_Y(y)\}.$$

Let A be a subset of R^ℓ, we will denote, respectively, by ∂A, $cl(A)$, $int(A)$ and $cone(A)$, the boundary of A, the closure of A, the interior of A and the convex cone generated by A. If A is a subspace of R^ℓ, we denote by $proj_A$, the orthogonal projection on A.

If f is a given real-valued function defined on some subset A of R^ℓ, we define the epigraph of f by

$$\mathscr{E}(f) = \{(x, \mu) \in A \times R:\ \mu \ge f(x)\}$$

and we say that f is lower semi-continuous (l.s.c.) if $\mathscr{E}(f)$ is closed.

For x and y in R^ℓ, we will write $x \ge y$ (resp. $x \gg y$) when $x_h \ge y_h$ (resp. $x_h > y_h$) for $h = 1, \ldots, \ell$. Finally, we define the following sets:

$$R^\ell_+ = \{x \in R^\ell:\ x \ge 0\}$$

and

$$R^\ell_{++} = \{x \in R^\ell:\ x \gg 0\}$$

and, for e in R^ℓ, we will denote by $e^\perp$ the set defined by

$$e^\perp = \{x \in R^\ell:\ xe = 0\}.$$

2. *Q*-Lipschitz functions

Let Q be a nonnecessarily convex cone with vertex 0 of $R^n \times R$. If q is an element of Q we will denote by q' and q'', respectively, in R^n and R, the unique pair such that $q = (q', q'')$.

Let f be a real-valued function defined on a given compact subset K of R^n, we will say that f is Q-Lipschitz on K if f is l.s.c. and satisfies

$$\forall x \in K,\ \ \forall q \in Q, \quad x + q' \in K \Rightarrow f(x) + q'' \ge f(x + q')$$

or equivalently, from a geometric point of view, if $\mathscr{E}(f)$ is closed and satisfies

$$[\mathscr{E}(f) + Q] \cap (K \times R) \subset \mathscr{E}(f).$$

Example 1. For k in R let us consider the cone Q_k defined by

$$Q_k = \{(x, \mu):\ \mu \ge k\|x\|\}.$$

It is easy to see that f is Q_k-Lipschitz on a given set if and only if f is k-Lipschitz on this set.

Example 2. If $Q=(-R_+)^n \times R_+$, then it is easy to see that f is Q-Lipschitz on a given set if and only if f is nondecreasing relatively to each coordinate on this set. Note that f is not assumed to be continuous.

For a given cone Q with vertex 0 and nonempty interior and a given subset K of R^n, let us define the set $Lip(Q,K)$ as the set of Q-Lipschitz functions on K. We will construct in the next a particular topology on this set and we will call it δ-topology.

Let us consider that $(e_1,\dots,e_{n+1})$ is the canonical basis of $R^n \times R$ endowed with the canonical scalar product. It is easy to see that $Lip(Q,K)=Lip(Q',K)$ where $Q'=cone[Q\cup\{e_{n+1}\}]$.

Now remark that if $Q'' \subset Q'$ then $Lip(Q',K) \subset Lip(Q'',K)$. Furthermore, since Q and then Q' have a nonempty interior, then there exists an element e' in Q' and a positive real number ε' such that $\|e'\|=1$ and $B(e',\varepsilon') \subset Q'$. We clearly have $e' \neq -e_{n+1}$, indeed if this were not the case then we should have, for all $(x,\mu)\in R^n \times R$, $f(x)-\mu \geq f(x)$. Furthermore, if we define e by $e=(e_{n+1}+\lambda e')/\|e_{n+1}+\lambda e'\|$ and ε by $\varepsilon=(\lambda/\|e_{n+1}+\lambda e'\|)\varepsilon'$, for λ sufficiently small, then we have $ee_{n+1}>0$ and $B(e,\varepsilon)\subset Q'$.

It suffices then to construct the δ-topology on $Lip(Q^e_\varepsilon,K)$ with $Q^e_\varepsilon = cone[B(e,\varepsilon)]$. In order to simplify the notations, we will denote by Q the set Q^e_ε.

Let f in $Lip(Q,K)$ and let us define the set $E(f)$ by $E(f)=\mathscr{E}(f)+Q$. It is clear then that we have $E(f)+Q\subset E(f)$ and, by definition of $Lip(Q,K)$, $E(f)\cap[K\times R]=\mathscr{E}(f)$.

Let us now consider the real-valued function λ_f on $e^\perp$ defined for y in $e^\perp$ by $\lambda_f(y)=\inf\{\lambda\in R\colon\ y+\lambda e\in E(f)\}$. We also consider the function Λ_f on $e^\perp$ with values in $R^n\times R$ defined for y in $e^\perp$ by $\Lambda_f(y)=y+\lambda_f(y)$. We have the following:

Lemma 3 (Bonnisseau and Cornet [2]). *The functions λ_f and Λ_f are well defined and*
(i) *Λ_f is a homeomorphism between $e^\perp$ and $\partial E(f)$, with inverse $proj_{e^\perp}|_{\partial E(f)}$,*
(ii) *λ_f is Lipschitz and $\partial\lambda_f(y)=\{p\in e^\perp\colon\ p-e\in N_{E(f)}(\Lambda_f(y))\}$ for all y in $e^\perp$.*

Since $E(f)+Q\subset E(f)$, then for all z in $\partial E(f)$, we have $Q\subset T_{E(f)}(z)$ and consequently $N_{E(f)}(z)\subset Q^\circ$. Furthermore, we have $B(e,\varepsilon)\subset Q$ which implies that $Q^\circ\subset\{p+\mu e\colon\ p\in e^\perp, \mu\leq -\varepsilon\|p\|\}$ and consequently $\partial\lambda_f(y)\subset\{p\in e^\perp\colon\ \|p\|\leq 1/\varepsilon\}$ for all y in $e^\perp$. The function λ_f is then $1/\varepsilon$-Lipschitz on $e^\perp$.

In the next, we will consider subsets of $Lip(Q,K)$ denoted by $Lip^M(Q,K)$ (with M in $R\cup\{+\infty\}$) and defined as the set of functions of $Lip(Q,K)$ bounded by M on K.

Lemma 4. *If f and g are two functions in $Lip^M(Q,K)$ with $\lambda_f(y)=\lambda_g(y)$ for all y in $proj_{e^\perp}(K\times[-M,M])$ then $f=g$.*

Proof. Let x in K, it is easy to see that $(x,f(x))\in\partial E(f)\cap(K\times[-M,M])$. If we denote by y the projection of $(x,f(x))$ on $e^\perp$, we have then $y+\lambda_f(y)e=(x,f(x))$ and consequently $y+\lambda_g(y)e=(x,f(x))$. This proves that $(x,f(x))\in E(g)$ and it is easy to show then that $g(x)$ is necessarily lower than $f(x)$. A symmetrical reasoning gives us that $f(x)=g(x)$ and then $f=g$ on K. $\square$

Definition 5. The δ-topology on $Lip^M(Q,K)$ is defined as the weakest topology for which the mapping $\Psi: f \to \lambda_f$ defined from $Lip^M(Q,K)$ into the set of $1/\varepsilon$-Lipschitz functions on $proj_{e^\perp}(K \times [-M,M])$ endowed with the uniform convergence topology is continuous.

In particular, we have that the sequence (f_n) converges to f in $Lip^M(Q,K)$ if and only if the sequence (λ_{f_n}) converges to λ_f for the uniform convergence topology on $proj_{e^\perp}(K \times [-M,M])$.

Theorem 6. *For $M<+\infty$, $Lip^M(Q,K)$ is compact for the δ-topology.*

Proof. Let f be in $Lip^M(Q;K)$ and let x_0 be an element of K. We have clearly that $(x_0,M)\in\mathscr{E}(f)$ and we can find a unique pair (z_1,z_2) in $e^\perp \times R$ such that $(x_0,M)=z_1+z_2e$. Let y be an element of $e^\perp$, let t be the real number defined by $t=2\|z_1-y\|/\varepsilon$ and let v be a vector of $e^\perp$ defined by $v=-(z_1-y)/2\|z_1-y\|$. It is easy to see that $y+(t+z_2)e=(x_0,M)+t(e+\varepsilon v)$. Since $e+\varepsilon v$ is in $B(e,\varepsilon)$ and (x_0,M) is in $\mathscr{E}(f)$, then $t(e+\varepsilon v)$ is in Q and $y+(t+z_2)e$ is in $E(f)$. Consequently, we have that $\lambda_f(y)\le t+z_2$ and can be bounded independently from y when y is in $proj_{e^\perp}(K\times[-M,M])$.

We have then that $\Psi[Lip^M(Q;K)]$ is bounded in the set of all the real valued $1/\varepsilon$-Lipschitz functions on $proj_{e^\perp}(K\times[-M,M])$. Following Ascoli's theorem, $\Psi[Lip^M(Q;K)]$ is then relatively compact. Since Ψ is, by definition of the δ-topology and by Lemma 2, a homeomorphism between $Lip^M(Q;K)$ and $\Psi[Lip^M(Q;K)]$ we have that $Lip^M(Q;K)$ is relatively compact. The closedness of this last set is easy to check and we obtain then the required result. $\square$

Next, we will establish links between the δ-topology and some classical notions of convergence in functional spaces.

Proposition 7. *If (f_n) is a sequence in $Lip^M(Q;K)$ converging for the δ-topology to some f in the same set then we have the pointwise convergence at every continuity point of f in $int(K)$.*

Proof. In order to simplify the notations, we will denote λ_{f_n} by λ_n and λ_f by λ. By definition of the δ-convergence, we have that the sequence λ_n converges uniformly to λ. Let x be in $int(K)$, let e' be the orthogonal projection of e on $(e_1,\ldots,e_n)$ and let $\alpha>0$ such that $x'=x-\alpha e'$ is in K. Let us consider now y in $e^\perp$ defined as the orthogonal projection of $(x',f(x'))$ on $e^\perp$ and γ such that $y+\gamma e=(x',f(x'))$. We can easily check that $\lambda(y)=\gamma$ and we can choose n_0 sufficiently large in order to have $\|\lambda_n(y)-\lambda(y)\|<\alpha$ for all $n\ge n_0$. By definition of λ_n, we have $y+\lambda_n(y)e+q\in E(f_n)$ for all q in Q. It suffices to consider $q=(\alpha-\lambda_n(y)+\lambda(y))e$ to obtain that $(x',f(x'))+\alpha e\in E(f_n)$ which clearly implies that $f_n(x)\le f(x')+\alpha ee_{n+1}$ or equivalently $f_n(x)\le f(x-\alpha e')+\alpha ee_{n+1}$.

Since f and f_n play the same role, we have also $f(x) \le f_n(x - \alpha e') + \alpha e e_{n+1}$ and if we replace x by $x + \alpha e'$ we obtain that $f(x + \alpha e') \le f_n(x) + \alpha e e_{n+1}$. If x is a continuity point of f then for a given $\eta > 0$ we can chose $\alpha < \eta$ in order to have $\|f(x + \alpha e') - f(x)\| \le \eta$ and $\|f(x - \alpha e') - f(x)\| \le \eta$. For such an α we have then that $\|f_n(x) - f(x)\| \le 2\eta$. This completes the proof. □

Proposition 8. *If the sequence (f_n) converges to f in $Lip^M(Q;K)$ for the δ-topology and if f is continuous on $int(K)$ then the sequence (f_n) converges uniformly to f on all the compact subsets of $int(K)$.*

Proof. The proof of this result is a direct adaptation of the previous one. Indeed on the compact subsets of $int(K)$, f is uniformly continuous and we can then chose α independently from x in order to have $x \pm \alpha e' \in int(K)$ and $\|f(x \pm \alpha e') - f(x)\| \le \delta$. □

Lemma 9. *If f and g are in $Lip^M(Q;K)$ with $f \ge g$, then $\int_K (f - g) \le \int_{proj_{e^\perp}(K \times [-M;M])} (\lambda_f - \lambda_g)$.*

Proof. It is easy to see that $\int_K f(x)\,\mathrm{d}x = \int_{-M \le y \le f(x); x \in K} \mathrm{d}x\,\mathrm{d}y - \int_{-M \le y \le 0; x \in K} \mathrm{d}x\,\mathrm{d}y$.

Let now $(x, y) \in K \times [-M; \infty[$ such that $y < f(x)$, we have clearly that $(x, y) \notin E(f)$ which is equivalent to $proj_{e^\perp}(x, y) + [(x, y)e]e \notin E(f)$ which in turn is equivalent to $(x, y)e < \lambda_f(proj_{e^\perp}(x, y))$.

Furthermore, we have already seen that when $y = f(x)$ then $(x, y) \cdot e = \lambda_f(proj_{e^\perp}(x, y))$. Let us denote by Δ and Δ', respectively, the following sets $\Delta = \{(x, y) \in K \times [-M, \infty[: (x, y)e < \lambda_f(proj_{e^\perp}(x, y))\}$ and $\Delta' = \{(x, y) \in K \times [-M, \infty[: (x, y)e \le \lambda_f(proj_{e^\perp}(x, y))\}$. It is easy to see that

$$\Delta \subset \{(x, y) \in K \times [-M, \infty[: y \le f(x)\} \subset \Delta'.$$

Let us denote by Δ'' the set $\{(x, y) \in K \times [-M, \infty[: y \le f(x)\}$, we have then

$$\int_{\Delta''} \mathrm{d}x\,\mathrm{d}y = \int_K f(x)\,\mathrm{d}x - M \int_K \mathrm{d}x$$

and, consequently,

$$\int_{\Delta} \mathrm{d}x\,\mathrm{d}y + M \int_K \mathrm{d}x \le \int_K f(x)\,\mathrm{d}x \le \int_{\Delta'} \mathrm{d}x\,\mathrm{d}y + M \int_K \mathrm{d}x.$$

Since we assumed that $f \ge g$ we have then $\int_K (f - g) \le \int_\Gamma \mathrm{d}x\,\mathrm{d}y$ where $\Gamma = \{(x, y) \in K \times [-M, \infty[: \lambda_g(proj_{e^\perp}(x, y)) < (x, y)e \le \lambda_f(proj_{e^\perp}(x, y))\}$.

Let us define the set Γ' by $\Gamma' = \{(u, v) \in proj_{e^\perp}(K \times [-M; M]) \times proj_e(K \times [-M; M]) : \lambda_g(u) \le v \le \lambda_f(u)\}$, it is clear then that we have $\int_K (f - g) \le \int_{\Gamma'} \mathrm{d}u\,\mathrm{d}v \le \int_{proj_{e^\perp}(K \times [-M;M])} (\lambda_f - \lambda_g)$. □

Proposition 10. *The δ-convergence implies the L^1 convergence.*

Proof. Let (f_n) be a sequence of functions converging in the sense of the δ-convergence to some function f. It is clear that

$$\int |f_n - f| = \int (|\tilde{f}_n - f_n| + |\tilde{f}_n - f|),$$

where $\tilde{f}_n = \max(f_n; f)$.

If we prove that $\tilde{\lambda}_n = \lambda_{\tilde{f}_n} = \max(\lambda_n, \lambda)$ then we have, by the previous lemma, that $\int_K |f_n - f| \leq \int_{proj_e^{\perp}(K \times [-M;M])} (\tilde{\lambda}_n - \lambda) + \int_{proj_e^{\perp}(K \times [-M;M])} (\tilde{\lambda}_n - \lambda_n)$. The uniform convergence of (λ_n) and $(\tilde{\lambda}_n)$ to λ permits then to conclude.

It only remains now to prove that $\tilde{\lambda}_n = \max(\lambda_n, \lambda)$. First, we can remark that $E(\tilde{f}_n) = E(f_n) \cap E(f)$. Assume now that we have, for some y in $e^{\perp}$, $\max(\lambda_n, \lambda)(y) = \alpha$. This implies that $y + \alpha e \in E(f_n)$ as well as $y + \alpha e \in E(f)$ and then $y + \alpha e \in E(\tilde{f}_n)$ which in turn imply that $\tilde{\lambda}_n(y) \leq \alpha$ and consequently $\tilde{\lambda}_n \leq \max(\lambda_n, \lambda)$.

Conversely, the inclusions $E(\tilde{f}_n) \subset E(f_n)$ and $E(\tilde{f}_n) \subset E(f)$ imply that $\tilde{\lambda}_n \geq \lambda_n$ and $\tilde{\lambda}_n \geq \lambda$ which completes the proof. □

3. Generalizations

In this section we will extend the previous concepts and results to non l.s.c. functions. First, it is easy to see without any proof that these results are verified if we consider u.s.c. functions instead of l.s.c. ones. Indeed, it suffices to consider $-f$ instead of f.

Let us now consider a function f from $A \subset R^n$ to R. We will say that f is C-continuous at some point x of A, for a given nonempty open cone C of R^n, if

$$\lim_{h \to 0, h \in C, x+h \in A} f(x+h) = f(x).$$

In particular, if $n = 1$ and $C = R_+$ (resp. R_-), the C-continuity is the right-continuity (resp. left-continuity).

Lemma 11. *If f and g are two C-continuous functions from $A \subset R^n$ to R with $cl[\mathscr{E}(f)] = cl[\mathscr{E}(g)]$, then $f = g$ on $int(A)$.*

Proof. Let f be a given function on A and let us define the function $\widetilde{f}$ by

$$\widetilde{f}(x) = \inf_{(x,y) \in cl[\mathscr{E}(f)]} y.$$

If f is C-continuous, then for all x_0 in $int(A)$, $\{x \in A\colon x - x_0 \in C\}$ is nonempty and we have

$$f(x_0) = \lim_{x \to x_0, x - x_0 \in C} \widetilde{f}(x).$$

Indeed, let $\{(x_n, \widetilde{f}(x_n))\}$ be a sequence converging to (x, ℓ) with $x_n - x \in C$ (and, in particular, $x_n - x \neq 0$) for all n and let, for each n, a sequence $\{(x_n^p, y_n^p)\}$ in $\mathscr{E}(f)$

converging to $(x_n, \widetilde{f}(x_n))$. By a diagonal extraction process, it is easy to see that there exists an application φ such that the sequence $\{(x_n^{\varphi(n)}, y_n^{\varphi(n)})\}$ converges to (x_0, ℓ). Since C is open, we can assume that the sequence $\{x_n^{\varphi(n)}\}$ is in $x_0 + C$, this ensures then that $f(x_0) = \ell$.

Now, since $\widetilde{f} = \widetilde{g}$ and are both C-continuous, we have clearly that $f = g$. □

Next, we will say that the function f is Q–C-Lipschitz, if it is C-continuous and satisfies

$$\forall x \in A,\ \ \forall q \in Q, \quad x + q' \in A \Rightarrow f(x) + q'' \geq f(x + q')$$

and we will denote by $Lip^M(Q, C, A)$ the set of Q–C-Lipschitz functions bounded by M on A.

Remark that the main difference between a Q–C-Lipschitz function and a Q-Lipschitz function is that the first one is C-continuous instead of l.s.c.

Since we have a one-to-one correspondence between the Q–C-Lipschitz functions on a given open set A and their epigraphs, the δ-topology can be extended to this space and we have the following.

Theorem 12. *Let $\{f_n\}$ be a sequence in $Lip^M(Q, C, A)$, where A is an open subset of R^n then, for each compact subset K of A, there exists a subsequence $\{f_{\varphi(n)}\}$ such that $\{f_{\varphi(n)}\}$ converges, for the δ-topology, on K.*

The proof of this result is analogous to the similar one for $Lip^M(Q, K)$. The main difference is that the one-to-one correspondence between the Q–C-Lipschitz functions on some set A and their epigraphs is well defined only if A is open. For this reason, in order to obtain the δ-convergence on K, we have to consider functions defined on an open set larger than K.

4. Applications

In this section we will consider some corollaries of the previous results and make some links with known results. Our aim is to show that many known results can be derived from our results which make then a synthesis between many different approaches in the literature.

The first result can be found in [6, Theorem 10.8]:

Corollary 13. *If (f_n) is a sequence of real convex C^1-functions defined on some compact interval $[a; b]$ of R and converging pointwise to some convex C^1 function f then (f_n) converges uniformly to f as well as (f_n') to f'.*

Proof. Let α be a sufficiently small positive real number and let $x \in [a + \alpha; b - \alpha]$. It is easy to see by a convexity argument that

$$\frac{f_n(a+\alpha) - f_n(a)}{\alpha} \leq f_n'(x) \leq \frac{f_n(b) - f_n(b-\alpha)}{\alpha},$$

and if n is sufficiently large in order to have $|f_n(y)-f(y)|\leq\alpha$ for $y=a, a+\alpha, b-\alpha, b$, we have

$$\frac{f(a+\alpha)-f(a)}{\alpha}-2\leq f_n'(x)\leq\frac{f(b)-f(b-\alpha)}{\alpha}+2.$$

The sequence (f_n') is then a sequence of nondecreasing and uniformly bounded functions on $[a+\alpha; b-\alpha]$ and by the compactness of this set of functions, there exists some nondecreasing function g on $[a+\alpha; b-\alpha]$ and some subsequence $\{f'_{\varphi(n)}\}$ converging to some function g.

Since g is nondecreasing, the set of discontinuity points of g has a zero measure and we have then by Proposition 4 the pointwise convergence of $(f'_{\varphi}(n))$ almost everywhere on $[a+\alpha; b-\alpha]$.

If $(f'_{\varphi(n)}(x))$ converges to $g(x)$ then for all $h>0$, we have (by a convexity argument),

$$\frac{f_{\varphi(n)}(x)-f_{\varphi(n)}(x-h)}{h}\leq f'_{\varphi(n)}(x)\leq\frac{f_{\varphi(n)}(x+h)-f_{\varphi(n)}(x)}{h}$$

and taking the limit when n goes to ∞ we obtain

$$\frac{f(x)-f(x-h)}{h}\leq g(x)\leq\frac{f(x+h)-f(x)}{h}.$$

If we take now the limit when h goes to zero we obtain then that $f'(x)=g(x)$ almost everywhere.

Since f' is a continuous nondecreasing function and g a nondecreasing function equal almost everywhere to f' we have then that $g=f'$ and g is continuous.

The continuity of g implies then, by Proposition 5 that $f'_{\varphi(n)}$ converges uniformly to $g=f'$. We have then that all the converging subsequences of (f_n') converge uniformly to f' and we can deduce that f_n' converges uniformly to f'. Since we have also the pointwise convergence of the sequence (f_n) to f, it is well known then that we have then the uniform convergence of (f_n) to f. $\square$

Let us now consider in R^k the vector $e=(1,\ldots,1)$. A real-valued function F on R^k will be said continuous from above at some point x if for each positive ε, there exists a positive η such that $x\leq y\leq x+\eta e$ implies $|F(x)-F(y)|<\varepsilon$. Recall now that a distribution function is a function $F(x)=F(x_1,\ldots,x_k)$ on R^k with the following three properties:

(i) F is everywhere continuous from above;

(ii) $0\leq F(x)\leq 1$ for all x, F is nondecreasing in each variable, and, for each k-dimensional rectangle $(a, b]$,

$$\sum\pm F(a_1+\theta_1 d_1,\ldots,a_k+\theta_k d_k)\geq 0,$$

where $d_i=b_i-a_i$, where the sum ranges over all 2^k sequences $(\theta_1,\ldots,\theta_k)$ of 0's and 1's, and where the sign $+$ or $-$, according as the number of 0's in the sequence, is even or odd;

(iii) $F(x) \to 0$ as any one coordinate of x goes to $-\infty$ and $F(x) \to 1$ as all coordinates of x go to ∞.

We can now prove the following result:

Corollary 14 (Helly's Selection Theorem). *If $\{F_n\}$ is a sequence of distribution functions on R^k, then there exists a subsequence $\{F_{\varphi(n)}\}$ and a function F satisfying conditions* (i) *and* (ii) *above* (*but perhaps not* (iii)) *such that*

$$\lim_n F_{\varphi(n)}(x) = F(x)$$

for all continuity points x of F.

Proof. It suffices to remark that a distribution function is in $Lip^1((-R_+)^n \times R_+, R^n_{++}, R^n)$ and we obtain then the existence, for each compact subset of R^n, of a δ-converging subsequence. By a diagonal extraction process we can construct a subsequence δ-converging to some function F for all compact subset of R^n. Since the δ-convergence implies the pointwise one for all continuity points of the limit and since conditions (i) and (ii) are inherited at the limit, this completes the proof. □

Let now Π be a family of probability measures on $(R^k, \mathscr{R}^k)$ where $\mathscr{R}^k$ is the class of Borel sets in R^k. If such probability measures P_n and P satisfy $\int_{R^k} f \, \mathrm{d}P_n \to \int_{R^k} f \, \mathrm{d}P$ for every bounded continuous real function f on R^n, we say that P_n converges weakly to P and write $P_n \Rightarrow P$. For a probability measure P in Π, we define the associated distribution function F by

$$F(x) = P\{y\colon\ y \leq x\}.$$

Such a function F satisfies the conditions (i)–(iii) introduced previously and it is well known (see [1], p.17) that $P_n \Rightarrow P$ if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at continuity points x of F.

We can now prove the following weak form of a theorem due to Prohorov:

Theorem 15 (Prohorov). *Every sequence of elements of Π contains a weakly-convergent subsequence if and only if for every positive ε there exists a compact set K such that $P(K) > 1 - \varepsilon$ for all P in Π* (*in this case, Π is said to be tight*).

Note that the strong form of Prohorov's Theorem claims that the "if" part is valid for every metric space S instead of R^k and the "only if" part for every separable and complete space S.

Proof. Assume that Π is tight and let us consider a sequence $\{P_n\}$ in Π with the associated sequence $\{F_n\}$. Following Helly's Theorem, there exists a subsequence $\{F_{\varphi(n)}\}$ contained in $\{F_n\}$ and a function F satisfying (i) and (ii) such that $\{F_{\varphi(n)}(x)\}$ converges to $F(x)$ at continuity points x of F. Let $\varepsilon > 0$ and K be a compact set such that $P(K) > 1 - \varepsilon$ and let us define the following constants:

$$M = \inf_{x \in K, h=1,\dots,k} x_h,$$

$$M' = \sup_{x \in K, h=1,\ldots,k} x_h.$$

If x has a coordinate lower than M, then we have $F_n(x) < \varepsilon$ for all n and if x has all its coordinates greater than M' then $F_n(x) > 1 - \varepsilon$ for all n. Since, following (ii), F is nondecreasing and consequently almost everywhere continuous, we obtain that $F(x) < \varepsilon$ in the first case and $F(x) > 1 - \varepsilon$ in the second one. This suffices to prove that F satisfies (iii) and is then associated to some probability measure P such that $P_{\varphi(n)} \Rightarrow P$.

Conversely, assume that Π is not tight. There exists then some $\varepsilon > 0$ and a sequence $\{P_n\}$ such that $P_n(B(0,n)) \leq 1 - \varepsilon$. Assume now that there exists a subsequence such that $P_{\varphi(n)} \Rightarrow P$ and let K be a compact set such that $P(K) > 1 - \varepsilon/2$. By Urysohn's Lemma, we can construct a continuous function f equal to 1 on K and equal to 0 out of some compact set K' containing K. Since $\int_{R^k} f \, \mathrm{d}P_{\varphi(n)} \to \int_{R^k} f \, \mathrm{d}P$, we obtain easily that $P_{\varphi(n)}(K') > 1 - \varepsilon$ for n sufficiently large which contradicts the definition of P_n. □

In fact for nondecreasing one-dimensional functions, our topology coincides with the topology defined by the Levy distance defined, for a pair (f, g) as the infimum of those positive ε such that

$$f(x - \varepsilon) - \varepsilon \leq g(x) \leq f(x + \varepsilon) + \varepsilon.$$

Geometrically this is the shortest distance between the graph of f and the graph of g along lines in the direction of the second diagonal (spanned by $(-1, 1)$). The choice of δ to name our topology is directly linked to this *diagonal* geometric property.

We will now try to apply our results in order to prove existence results for some differential equations.

Let Φ and Ψ two nonnegative continuous functions from $R \times R$ to R and assume that for x sufficiently large (larger than a given M) Φ is positive and greater than $K\Psi$ where K is a given constant.

Consider now the following differential equation on $[0, T]$:

$$\dot{x}\Phi(t, x) = \Psi(t, x)$$

with the following initial condition: $x(0) = x_0 \geq 0$.

If Φ is positive, we can construct the following mapping from the set $C([0, T], R)$ endowed with the uniform convergence topology to itself:

$$\Gamma : x(.) \to \Gamma(x)(.) = x_0 + \int_0^{\cdot} \frac{\Psi(s, x(s))}{\Phi(s, x(s))} \, \mathrm{d}s$$

we can easily check that Γ is a continuous mapping and that, for all nonnegative function x, $\Gamma(x)$ is nonnegative, bounded above by $x_0 + T(\max_{[0,T] \times [0,M]}(\Psi(t, y)/\Phi(t, y)) + (1/K))$ and $(\max_{[0,T] \times [0,M]}(\Psi(t, y)/\Phi(t, y)) + (1/K))$-Lipschitz.

By a fixed point argument as in Peano's Theorem, we prove that there exists a solution to our differential equation on $[0, T]$.

The following theorem permits to extend this result to a nonnecessarily positive function Φ.

Theorem 16. *There exists a left-continuous function* x *on* $[0,T]$ *such that* $x(0)=x_0$ *and*

$$\dot{x}(t)\Phi(t,x(t))=\Psi(t,x(t))$$

on every interval on which $\Phi(t,x(t))$ *is positive.*

Proof. Let us denote by x_n, the solution of the previous equation when Φ is replaced by $\Phi+(1/n)$, it is easy to check that x_n is an increasing function bounded below by x_0 and above by $x_0+T(\max_{[0,T]\times[0,M]}(\Psi(t,y)/\Phi(t,y))+(1/K))$.

Then there exists a subsequence δ-converging to some nondecreasing function $\hat{x}$. Note that, if Φ is bounded below by some positive constant C on a given interval, it is easy to see that the sequence $\{x_n\}$ is $(\max_{[0,T]\times[0,M]}(\Psi(t,y)/C)+(1/K))$-Lipschitz on that interval and then that there exists a subsequence of $\{x_n\}$ converging for the uniform convergence topology to $\hat{x}$ on that interval. Let us now define a left-continuous nondecreasing function x as follows:

$$x(t)=\sup\{\hat{x}(t-h)\colon h>0 \text{ and } t-h\geq 0\}$$

and $x(0)=x_0$.

If $\Phi(t_0,x(t_0))$ is positive then there exists, by continuity of Φ, a positive real number α such that $\Phi(t,y)>\frac{1}{2}\Phi(t_0,x(t_0))$ for all pair (t,y) such that $|t-t_0|<\alpha$ and $|y-x(t_0)|<\alpha$. Since x is left-continuous, there exists $\beta<\alpha$ such that for all t satisfying $t_0-\beta<t\leq t_0$ we have $\Phi(t,x(t))>\frac{1}{2}\Phi(t_0,x(t_0))$ and $|x(t)-x(t_0)|<\alpha/2$.

Since $\hat{x}(t)$ is almost everywhere equal to the limit of $\{x_n(t)\}$ then it is easy, by a diagonal extraction process, to construct sequences $\{t_n\}$ and $\{t_n'\}$ converging from below, respectively, to t_0 and $t_0-\beta$ and such that $x_n(t_n)$ and $x_n(t_n')$ converges, respectively, to $x(t_0)$ and $x(t_0-\beta)$.

Let, now $\varepsilon<\beta$ be a given positive real number. For n sufficiently large, we have $|x_n(t_n)-x(t_0)|<\alpha/2$, $t_0-\varepsilon<t_n\leq t_0$, $|x_n(t_n')-x(t_0-\beta)|<\alpha/2$ and $t_0-\beta-\varepsilon<t_n'\leq t_0-\beta$.

These inequalities imply that for $t\in(t_0-\beta;t_0-\varepsilon)$, we have $x(t_0-\beta)-(\alpha/2)<x_n(t)<x(t_0)+(\alpha/2)$ and then $x(t_0)-\alpha<x_n(t)<x(t_0)+(\alpha/2)$. Consequently, for n sufficiently large and for $t\in(t_0-\beta;t_0-\varepsilon)$ we have $\Phi(t,x_n(t))>\frac{1}{2}\Phi(t_0,x(t_0))$. As we have already seen, when Φ is bounded below the sequence x_n converges uniformly and the limit $\hat{x}$ is then continuous and consequently equal to x on $(t_0-\beta;t_0-\varepsilon)$.

This permits to write that for $t\in(t_0-\beta;t_0-\varepsilon)$,

$$x(t)-x(t_0-\beta)=\int_{t_0-\beta}^{t}\frac{\Psi(s,x(s))}{\Phi(s,x(s))}\,\mathrm{d}s$$

and this for all ε sufficiently small or equivalently

$$\dot{x}(t)\Phi(t,x(t))=\Psi(t,x(t))$$

on $(t_0-\beta;t_0)$ and then on all intervall where $\dot{x}(t)\Phi(t,x(t))$ is positive. $\square$

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