Comonotonic Processes

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Abstract

We consider in this paper two Markovian processes X and Y, solutions of a stochastic differential equation with jumps, that are comonotonic, i.e., that are such that for all t, almost surely, X_t is greater in one state of the world than in another if and only if the same is true for Y_t. This notion of comonotonicity can be of great use for mathematical finance issues. We show here that the assumption of comonotonicity imposes strong constraints on the coefficients of the diffusion part of X and Y.

1 Introduction

It is well-known that, in a complete financial market, there is a unique risk-neutral probability. However in incomplete markets, there is more than one such a risk-neutral measure and derivatives pricing becomes more complex. Perrakis and Ryan (1984) and Perrakis (1986, 1993) propose an ordering principle on the probabilities used to price the derivative asset. More precisely they restrict their attention to probability measures that are in reverse order than the stock price and derive derivatives pricing bounds using this particular set of probabilities. In Bizid, Jouini and Koehl (1999) and Jouini and Napp (2002) this ordering principle concerns the densities instead of the probabilities and is derived from the equilibrium theory, and more precisely from the market clearing conditions. They call "equilibrium bounds" the bounds obtained by assuming that stock returns and risk-neutral density are in reverse order and show that these bounds are tighter than the classical arbitrage bounds.

In order to illustrate this point, let us consider the simplest example of incomplete markets and let us explain the main idea of the so-called equilibrium interval. More precisely, consider a simple one-period model where the sample space and the probability are $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $P = (1/3, 1/3, 1/3)$.

First, there are two assets in the market: a first asset whose prices are $p(0) = 25$, at date 0 and $p(\omega_1) = 20, p(\omega_2) = 30$ or $p(\omega_3) = 40$ at date 1

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and a risk-free asset defined by an interest rate $r = \frac{1}{19}$. It is straightforward that the feasible range of the density of the martingale measures with respect to P is $(Q(\omega_1), Q(\omega_2), Q(\omega_3)) = (0.368 + Q(\omega_3), 0.632 - 2Q(\omega_3), Q(\omega_3))$ where $Q(\omega_3) \in [0, 0.316]$.

If there is a representative agent in this economy, the maximization program of this agents leads to the following first-order conditions:

$$(Q(\omega_1), Q(\omega_2), Q(\omega_3)) = (\frac{u'(20)}{u'(25)}, \frac{u'(30)}{u'(25)}, \frac{u'(40)}{u'(25)})$$

and since u is usually strictly-concave, we must have

$$Q(\omega_1) > Q(\omega_2) > Q(\omega_3).$$

After imposing these restrictions, we get the feasible range of the risk-neutral probability $Q(\omega_3) \in [0.088, 0.211]$. Consequently this approach permits to obtain bounds on the price of the derivative assets, which are better than the usual arbitrage-free bounds.

The aim of this paper is to examine in continuous time and in a general Levy-processes setting this "comonotonicity condition" and to prove that it imposes very strong constraints on the model parameters. This comonotonicity tool was also used in Borch (1962) and in Wilson (1968) to characterize Pareto optimal allocations of random endowments. It has enriched the domain of inequality measurement through the papers of Weymark (1981) and Yaari (1988). Landsberger and Meilijson (1994) mention these results and apply them to risk sharing through insurance. They also exhibit the link between the dispersion order defined by Bickel and Lehmann (1979) and comonotonicity. The comonotonicity concept is also used by Dybvig (1988) in order to characterize efficient strategies in financial markets.

We want to show that the assumption of comonotonicity for two processes imposes strong constraints on the coefficients of the diffusion part of the processes.

We start by introducing the notion of comonotonicity. We shall define it first for random variables and then for stochastic processes.

Definition 1 *Two real-valued random variables x_1 and x_2 defined on the same probability space (Ω, F, P) are comonotonic if there exists A in F, with probability one, and such that*

$$[x_1(\omega) - x_1(\omega')][x_2(\omega) - x_2(\omega')] \geq 0 \quad \textit{for all } (\omega, \omega') \in A \times A.$$

Notice that if two random variables x_1 and x_2 are such that there exists a nondecreasing function φ for which x_1 can be written in the form $x_1 = \varphi(x_2)$ (or if x_2 can be written in the form $x_2 = \varphi(x_1)$), then x_1 and x_2 are comonotonic. In fact, x_1 and x_2 are comonotonic if and only if they are nondecreasing functions of the same third random variable x_3. In particular, x_3 can be chosen to be equal to $x_1 + x_2$ (Denneberg (1994), Proposition 4.5, p.54).

Definition 2 *Two real-valued adapted processes X^1 and X^2 defined on the same filtered probability space $\left(\Omega, F, (F_t)_{t\geq 0}, P\right)$ are comonotonic if for all $t \geq 0$, the random variables X_t^1 and X_t^2 are comonotonic.*

Notice that if two processes X^1 and X^2 are such that for all t, $X_t^1 = d\left(t, X_t^2\right)$ where for all t, $d(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is some nondecreasing function, then X^1 and X^2 are comonotonic.
Besides, if d is of class $C^{1,2}$ and $X = \left(X^1, X^2\right)$ is a diffusion process of the form

$$dX_t = b_t dt + \sigma_t dW_t$$

where the $\mathbb{R}^2$-valued process $b \equiv \left(b^{X^1}, b^{X^2}\right)^*$, as well as the matrix-valued process $\sigma \equiv \left(\sigma^{X^1}, \sigma^{X^2}\right)^*$, where $\sigma^{X^1} \equiv (\sigma_1, \sigma_2)$ and $\sigma^{X^2} \equiv (\sigma_3, \sigma_4)$, satisfy the usual regularity conditions, then the use of Itô's Lemma enables us to get that

$$\begin{aligned} dX_t^1 &= \left\{ d_t\left(t, X_t^2\right) + d_x\left(t, X_t^2\right) b_t^{X^2} + 1/2 d_{xx}\left(t, X_t^2\right) \left|\sigma_t^{X^2}\right|^2 \right\} dt \\ &\quad + d_x\left(t, X_t^2\right) \sigma_t^{X^2} dW_t. \end{aligned}$$

Identifying the diffusion parts, we immediately obtain that for all t,

$$\sigma_t^{X^1} = \sigma_t^{X^2} d_x\left(t, X_t^2\right) \tag{1}$$

so that for all t,

$$\det \sigma(t) = \sigma_1(t)\sigma_4(t) - \sigma_3(t)\sigma_2(t) = 0 \qquad P \text{ a.s.}$$

In the general diffusion case[1], remark that if X^1 and X^2 are comonotonic, then the law of $\left(X^1, X^2\right)$ is singular with respect to the Lebesgue measure. The problem can be treated as follows. Let $T_a \equiv \inf\left\{t, \det \sigma_t \sigma_t^* > a\right\}$. The pair $\left(X_t^1, X_t^2\right)$ is a non-homogeneous diffusion process with transition kernels $P_{s,t}$ and as soon as $\det \sigma_t \sigma_t^* \neq 0$ and σ is continuous, then $P_{s,t}(x, \cdot)$ admits a density with respect to the Lebesgue measure for all t in an interval $[s, s+\varepsilon]$. Since $E\left[f(X_t)\right] \geq E\left[P_{T_a, t-T_a} f\left(X_{T_a}\right) 1_{\{T_a < t\}}\right]$ for any nonnegative f, it follows that the joint law of $\left(X_t^1, X_t^2\right)$ is not singular with respect to the Lebesgue measure for some t, as soon as $P(T_a < \infty) > 0$. Hence, if X^1 and X^2 are comonotonic, then $P(T_a = \infty) = 1$ for all $a > 0$, that is $\det \sigma_t \sigma_t^* = 0$ for all t.

We want to get an analogous result in the general case of two processes which are solutions of a stochastic differential equation with jumps. Remark that in the case where one of them can be written as a regular function of the other, then, as above, Itô's Lemma concludes.

Let (Ω, F, P) be a given probability space and $(F_t)_{t\geq 0}$ denote a right-continuous, complete filtration. Let $W = \left\{\left(W_t^1, ..., W_t^d\right)^*; t \geq 0\right\}$ denote a d-dimensional

[1] We are grateful to an anonymous referee for providing this short proof in the diffusion case.

Brownian motion for $(F_t)_{t\geq 0}$. Let $\mathcal{M}$ denote the set of real valued $(2\times d)$ -matrices.
Let ν be a finite measure on $\mathbb{R}^k$. Let $\sigma:\mathbb{R}^2\to\mathcal{M}$ and $b:\mathbb{R}^2\to\mathbb{R}^2$ and $f:\mathbb{R}^2\times\mathbb{R}^k\to\mathbb{R}^2$ be Borel measurable functions such that for some positive constants A and K ,

$$|\sigma(x)-\sigma(y)|^2+|b(x)-b(y)|^2+\int_{B(0;1)}|f(x,u)-f(y,u)|^2\,n(du)\leq K|x-y|^2 \tag{2}$$

$$|\sigma(x)|^2+|b(x)|^2+|f(x,u)|^2\leq A^2 \tag{3}$$

for x,y in $\mathbb{R}^2$ and u in $\mathbb{R}^k$ where as usual, for $m\in\mathcal{M}$ given by $m=\begin{pmatrix} m_{11} & \dots & m_{1d}\\ m_{21} & \dots & m_{2d}\end{pmatrix}$, we let $|m|\equiv\sqrt{\sum_{i,j}m_{ij}^2}$ and for $x\in\mathbb{R}^N$ given by $x=(x_1,...,x_N)^*$, $|x|\equiv\sqrt{\sum_{i=1}^N x_i^2}$.

Let μ be the Poisson measure on $\mathbb{R}_+\times\mathbb{R}^k$ with intensity $ds\otimes\nu(du)$ and $\widetilde{\mu}=\mu-ds\otimes\nu(du)$ its compensated measure. We suppose that μ is independent of the Brownian motion W. Let p be the (F_t)-stationary Poisson point process associated with the counting measure μ (see e.g. Ikeda-Watanabe (1981), Section II-3). Under our conditions, we know that the following stochastic differential equation

$$\begin{aligned} X_t &= X_0+\int_0^t b(X_s)\,ds+\int_0^t\sigma(X_s)\,dW_s+\int_0^t\int_{|z|>1}f(X_{s-},z)\,\mu(ds,dz)\\ &\quad+\int_0^t\int_{|z|\leq 1}f(X_{s-},z)\,\widetilde{\mu}(ds,dz) \end{aligned}\tag{4}$$

with given initial condition $X_0=\left(X_0^1,X_0^2\right)$, where X_0 is supposed to be a square integrable $\mathbb{R}^2$-valued F_0 -measurable random variable, admits a unique $(F_t)_{t\geq 0}$-adapted, càdlàg 2-dimensional solution process. We shall in the remainder of the paper write indifferently $\sigma(X_t)$ (resp. $b(X_t)$) or σ_t (resp. b_t).

In such a framework, we shall prove the following

Theorem 1 *If the two-dimensional solution process X of Equation* (4) *has comonotonic components X^1 and X^2, then for all $t\geq 0$, its dispersion matrix σ_t almost surely does not have full rank.*

2 Proof of Theorem 1

To prove Theorem 1, we shall assume that there exists $t_0\geq 0$ such that the dispersion matrix σ_{t_0} has full rank with a positive probability and show that the two processes X^1 and X^2 cannot be comonotonic. The rough idea is that if the dispersion matrix has full rank, then according to the fact that $W=\left(W^1,...,W^d\right)^*$ is a d-dimensional Brownian motion, the processes ΔX^1 and ΔX^2 do not necessarily have a "parallel" evolution[2] and as long as we take $X_{t_0}^1$

[2] For any process $Y=\{Y_t;t\geq t_0\}$, let ΔY denote the stochastic process $\{Y_t-Y_{t_0};t\geq t_0\}$.

and $X^2_{t_0}$ in a small enough interval, we will be able to find $\Delta t \equiv t - t_0 \geq 0$ such that the two random variables $X^1_{t_0+\Delta t}$ and $X^2_{t_0+\Delta t}$ are not comonotonic.

In Section 2.1, we exhibit an event B_{t_0} in F_{t_0} on which the dispersion matrix σ_{t_0} has full rank and each of the random variables $X^1_{t_0}$, $X^2_{t_0}$ and $\sigma_{ij}(t_0)$ for $i = 1, 2$ and $j = 1, ..., d$ is stuck in an interval of given length. In Section 2.2, we show that on some subevents, the problem can be reduced to the one with constant coefficients and a diffusion process. In Section 2.3, we prove that these events have a positive probability and we conclude.

2.1 A Specific Set at $t = t_0$

Suppose that for some $t_0 \in \mathbb{R}^*_+$, $\det \sigma_{t_0}\sigma^*_{t_0} \neq 0$ with a positive probability. Without loss of generality, we can assume that $\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0) \neq 0$ with a positive probability. We show that there exists an event B_{t_0} in F_{t_0}, with positive probability, on which each of the random variables $X^1_{t_0}$, $X^2_{t_0}$ and $\sigma_{ij}(t_0)$ for $i = 1, 2$ and $j = 1, ..., d$ is stuck in an interval of given length and on which $\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0) \neq 0$.

To do so, consider first $B \equiv \{\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0) \neq 0\}$. By assumption, we have $P(B) \neq 0$. Then there exists $\ell \in \mathbb{R}^*_+$ such that the event B_0 given by

$$B_0 \equiv \{|\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0)| \geq \ell\}$$

is of positive probability. Moreover, we can assume that the sign of the expression $\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0)$ remains constant on B_0.

Let $n \in \mathbb{N}$. Let for all k in $\mathbb{Z}$, for $i = 1, 2$ and $j = 1, ..., d$ and $l = 1, 2$,

$$C^{i,j}_k \equiv \left\{\sigma_{ij}(t_0) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right\}$$
$$D^l_k \equiv \left\{X^l_{t_0} \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right[\right\}$$

As

$$B_0 = \cup_{\substack{k_{i,j}\in\mathbb{Z} \\ k'_l\in\mathbb{Z}}} \left[B_0 \underset{i=1,2;j=1,...,d}{\cap} C^{i,j}_{k_{i,j}} \underset{l=1,2}{\cap} D^l_{k'_l}\right]$$

there exist $k_{i,j}$ for $i = 1, 2, j = 1, ..., d$ and k'_1, k'_2 in $\mathbb{Z}$ such that the event B_{t_0} given by $B_{t_0} \equiv B_0 \underset{i=1,2;j=1,...,d}{\cap} C^{i,j}_{k_{i,j}} \underset{l=1,2}{\cap} D^l_{k'_l}$ has positive probability. It is immediate that B_{t_0} satisfies the conditions mentioned above, the length of the intervals being equal to $1/2^n$. We consider a decreasing sequence of such nested sets $B_{t_0}(n)$. Since $(\sigma_{ij}(t_0))_{i=1,2;j=1,...,d}$ is stuck in a compact set and $|\sigma_{11}(t_0)\sigma_{22}(t_0) - \sigma_{21}(t_0)\sigma_{12}(t_0)| \geq \ell$, there exists some n_0, such that for all n greater than n_0, $a_{11}a_{22} - a_{21}a_{12} \neq 0$ holds true for any a_{ij} in $\left[\frac{k_{ij}}{2^n}, \frac{k_{ij}+1}{2^n}\right[$. For such an n_0, we let $\bar{\sigma}_i \equiv \frac{k_{ij}+1}{2^{n_0}}$ and $\underline{\sigma}_i \equiv \frac{k_{ij}}{2^{n_0}}$.

2.2 An Intermediary Lemma

We shall denote by $\tilde{X}$ the stochastic process $\left\{\tilde{X}_t = \left(\tilde{X}_t^1, \tilde{X}_t^2\right)^*; t \geq t_0\right\}$ given by

$$\widetilde{X}_t = X_{t_0} + \sigma_{t_0} \Delta W_t$$

and by Z the stochastic process $\left\{Z_t = \left(Z_t^1, Z_t^2\right)^*; t \geq t_0\right\}$ given by

$$\begin{aligned} Z_t &= \int_{t_0}^t b_s ds + \int_{t_0}^t (\sigma_s - \sigma_{t_0}) \, dW_s + \int_{t_0}^t \int_{|z|>1} f(X_{s-}, z) \, \mu(ds, dz) \\ &\quad + \int_{t_0}^t \int_{|z|\leq 1} f(X_{s-}, z) \, \widetilde{\mu}(ds, dz) \end{aligned}$$

Then for all $t \geq t_0$, $X_t = \tilde{X}_t + Z_t$ and $\Delta X = \Delta \tilde{X} + \Delta Z$.

Finally, for a given $\eta \in R_+^*$, let Z^η be given by

$$\begin{aligned} Z_t^\eta &= \int_{t_0}^t b_s ds + \int_{t_0}^t \varphi^\eta (\sigma_s - \sigma_{t_0}) \, dW_s + \int_{t_0}^t \int_{|z|>1} f(X_{s-}, z) \, \mu(ds, dz) \\ &\quad + \int_{t_0}^t \int_{|z|\leq 1} f(X_{s-}, z) \, \widetilde{\mu}(ds, dz) \end{aligned}$$

where $\varphi^\eta(x)$ stands for $x 1_{|x|\leq\eta} + \frac{x}{|x|} 1_{|x|>\eta}$.

Using the Lipschitz condition[3] on σ, we know that for all given $\eta \in \mathbb{R}_+^*$, there exists a positive real number $\varepsilon(\eta)$ such that for all x and y in $\mathbb{R}^2$ satisfying $|x - y| \leq \varepsilon(\eta)$,

$$|\sigma(x) - \sigma(y)| \leq \eta$$

For all $(\lambda, \Delta t, \eta, n) \in \left(\mathbb{R}_+^*\right)^3 \times \mathbb{N}$, we let $B^1_{\lambda,\Delta t,\eta,n}$ denote the set

$$\left\{\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right\} \cap \left\{\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)\right\} \cap \left\{\begin{array}{c} \Delta \tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda \\ \Delta \tilde{X}^2_{t_0+\Delta t} \leq -\lambda \end{array}\right\}$$

and $B^2_{\lambda,\Delta t,\eta,n}$ denote the set

$$\left\{\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right\} \cap \left\{\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)\right\} \cap \left\{\begin{array}{c} \Delta \tilde{X}^1_{t_0+\Delta t} \leq -\lambda \\ \Delta \tilde{X}^2_{t_0+\Delta t} \geq \frac{1}{2^n} + \lambda \end{array}\right\}$$

For $l = 1, 2$, we let $A^l_{\lambda,\Delta t,\eta,n} \equiv B^l_{\lambda,\Delta t,\eta,n} \cap B_{t_0}(n)$ and we prove the following

Lemma 1 *If there exist $[(\lambda_1, \eta_1), (\lambda_2, \eta_2)] \in \left(R_+^*\right)^2 \times \left(R_+^*\right)^2$, $\Delta t \in \mathbb{R}_+^*$, and $n \in \mathbb{N}$ for which $P\left[A^l_{\lambda_i,\Delta t,\eta_i,n}\right] > 0$ for $l = 1, 2$, then the two processes X^1 and X^2 cannot be comonotonic.*

[3] In fact, we only need a uniform continuity condition.

Proof Let us see first what happens on $A^1_{\lambda,\Delta t,\eta,n}$: we have $\sup_{s\in[t_0,t_0+\Delta t]}|\Delta X_s| \leq \varepsilon(\eta)$ hence for all $s\in[t_0,t_0+\Delta t]$

$$|\sigma_s - \sigma_{t_0}| \leq \eta$$

so that for all $s\in[t_0,t_0+\Delta t]$, $Z_s = Z^\eta_s$ and $|\Delta Z_{t_0+\Delta t}| = \left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda$.
As $\Delta X = \Delta\tilde{X}+\Delta Z$, we get on $A^1_{\lambda,\Delta t,\eta,n}$ that $\Delta X^1_{t_0+\Delta t} = \Delta\tilde{X}^1_{t_0+\Delta t}+\Delta Z^1_{t_0+\Delta t} \geq \frac{1}{2^n}$ and $\Delta X^2_{t_0+\Delta t} \leq 0$.

Now, using the same method, we get that for all $(\lambda,\Delta t,\eta,n) \in \left(\mathbb{R}^*_+\right)^3 \times \mathbb{N}$, we have $\Delta X^1_{t_0+\Delta t} \leq 0$ and $\Delta X^2_{t_0+\Delta t} \geq \frac{1}{2^n}$ on $A^2_{\lambda,\Delta t,\eta,n}$.

As $X^1_{t_0}$ and $X^2_{t_0}$ both belong to a (semi-open) interval of given length equal to $\frac{1}{2^n}$ on $A^l_{\lambda_l,\Delta t,\eta_l,n}$, we get that for all $(\omega,\omega') \in A^1_{\lambda_1,\Delta t,\eta_1,n} \times A^2_{\lambda_2,\Delta t,\eta_2,n}$, $X^1_{t_0+\Delta t}(\omega) > X^1_{t_0+\Delta t}(\omega')$ whereas $X^2_{t_0+\Delta t}(\omega) < X^2_{t_0+\Delta t}(\omega')$ so that

$$\left[X^1_{t_0+\Delta t}(\omega) - X^1_{t_0+\Delta t}(\omega')\right] \times \left[X^2_{t_0+\Delta t}(\omega) - X^2_{t_0+\Delta t}(\omega')\right] < 0$$

for all $(\omega,\omega') \in A^1_{\lambda_1,\Delta t,\eta_1,n} \times A^2_{\lambda_2,\Delta t,\eta_2,n}$, and the two random variables $X^1_{t_0+\Delta t}$ and $X^2_{t_0+\Delta t}$ cannot be comonotonic, which completes the proof of the lemma.□

So the lemma reduces the proof of our theorem to the finding of $[(\lambda_l,\eta_l)]_{l=1,2} \in \left(\mathbb{R}^*_+\right)^2 \times \left(\mathbb{R}^*_+\right)^2$, $n\in\mathbb{N}$ and $\Delta t \in \mathbb{R}^*_+$ for which the two events $A^1_{\lambda_1,\Delta t,\eta_1,n}$ and $A^2_{\lambda_2,\Delta t,\eta_2,n}$ have positive probability.

2.3 End of the Proof of Theorem 1

We consider first the set $A^1_{\lambda,\Delta t,\eta,n}$ and we only need to show that there exist $(\lambda,\Delta t,\eta,n) \in \left(\mathbb{R}^*_+\right)^3 \times \mathbb{N}$ for which

$$\begin{aligned} & P\left\{\left(\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right) \cap B_{t_0}\right\} + P\left\{\left(\sup_{s\in[t_0,t_0+\Delta t]}|\Delta X_s| \leq \varepsilon(\eta)\right) \cap B_{t_0}\right\} \\ & +P\left\{\begin{pmatrix} \Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n}+\lambda \\ \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda \end{pmatrix} \cap B_{t_0}\right\} \\ > \ & 2P(B_{t_0}). \end{aligned} \tag{5}$$

We first consider the set $\left\{\begin{pmatrix} \Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n}+\lambda \\ \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda \end{pmatrix} \cap B_{t_0}\right\}$. We shall denote by $\hat{X}$ the stochastic process $\left\{\hat{X}_t = \left(\hat{X}^1_t, \hat{X}^2_t\right)^*; t\geq t_0\right\}$ given by

$$\hat{X}_t = X_{t_0} + a^0\Delta W_t$$

where $a^0 \equiv \begin{pmatrix} a^0_{11} & a^0_{12} & 0 & ... & 0 \\ a^0_{21} & a^0_{22} & 0 & ... & 0 \end{pmatrix}$ for some real numbers $a^0_{ij} \in \left[\frac{k_{ij}}{2^n}, \frac{(k_{ij}+1)}{2^n}\right[$ for $i,j=1,2$. Then

$$\Delta\tilde{X}_{t_0+\Delta t} = \Delta\hat{X}_{t_0+\Delta t} + \left[\sigma_{t_0} - a^0\right]\Delta W_{t_0+\Delta t}$$

On $B_{t_0}(n)$, $\sigma_{ij}(t_0) \in \left[\frac{k_{ij}}{2^n}, \frac{k_{ij}+1}{2^n}\right[$, so that $\left|\sigma_{ij}(t_0) - a^0_{ij}\right| < \frac{1}{2^n}$. It is easy to see that for a given positive real number ξ, if $\Delta\hat{X}^1_{t_0+\Delta t} \geq 2\lambda+\xi$, $\Delta\hat{X}^2_{t_0+\Delta t} \leq -2\lambda-\xi$, $\left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\lambda 2^n-1}{2}$ for $j=1,2$, $\left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\xi}{(d-2)A}$ for $j=3,...,d$, then $\Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n}+\lambda$ and $\Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda$. So

$$
\begin{aligned}
& P\left[B_{t_0} \cap \left\{\Delta\tilde{X}^1_{t_0+\Delta t} \geq \frac{1}{2^n}+\lambda; \Delta\tilde{X}^2_{t_0+\Delta t} \leq -\lambda\right\}\right] \\
\geq\ & P\left[B_{t_0} \cap \left\{\begin{array}{c} \Delta\hat{X}^1_{t_0+\Delta t} \geq 2\lambda+\xi \\ \Delta\hat{X}^2_{t_0+\Delta t} \leq -2\lambda-\xi \end{array}; \begin{array}{c} \left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\lambda 2^n-1}{2},\ j=1,2 \\ \left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\xi}{(d-2)A},\ j=3,...,d \end{array}; \right\}\right] \\
\geq\ & \frac{P(B_{t_0})\, P\left(B^\xi\right)}{2\pi\Delta t} \int_{\substack{a^0_{11}x+a^0_{12}y\geq 2\lambda+\xi \\ a^0_{21}x+a^0_{22}y\leq -2\lambda-\xi \\ |x|\leq \frac{\lambda 2^n-1}{2}, |y|\leq \frac{\lambda 2^n-1}{2}}} e^{-\frac{x^2+y^2}{2\Delta t}}\, dxdy
\end{aligned}
$$

where $B^\xi = \left\{\left|\Delta W^j_{t_0+\Delta t}\right| \leq \frac{\xi}{(d-2)A}, j=3,...,d\right\}$ because μ and W are independent and independent of F_{t_0}.

Let us now consider the other sets involved in Inequality (5), i.e., the sets $B_{t_0} \cap \left\{\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right\}$ and $B_{t_0} \cap \left\{\sup_{s\in[t_0,t_0+\Delta t]} |\Delta X_s| \leq \varepsilon(\eta)\right\}$.
As for Z^η, we have

$$
\begin{aligned}
P\left[\left\{\left|\Delta Z^\eta_{t_0+\Delta t}\right| \leq \lambda\right\}\right] \geq\ & 1 - P\left[\left|\int_{t_0}^t b_s ds\right| > \frac{\lambda}{4}\right] - P\left[\left|\int_{t_0}^t \varphi^\eta(\sigma_s - \sigma_{t_0})\, dW_s\right| > \frac{\lambda}{4}\right] \\
& - P\left[\left|\int_{t_0}^t \int_{|z|>1} f(X_s,z)\, \mu(ds,dz)\right| > \frac{\lambda}{4}\right] \\
& - P\left[\left|\int_{t_0}^t \int_{|z|\leq 1} f(X_s,z)\, \widetilde{\mu}(ds,dz)\right| > \frac{\lambda}{4}\right]
\end{aligned}
$$

By Itô's isometry, we get

$$
\begin{aligned}
P\left[\left|\int_{t_0}^t \varphi^\eta(\sigma_s - \sigma_{t_0})\, dW_s\right| > \frac{\lambda}{4}\right] &\leq \frac{16}{\lambda^2} E\left[\left|\int_{t_0}^t \varphi^\eta(\sigma_s - \sigma_{t_0})\, dW_s\right|^2\right] \\
&\leq \frac{32\eta^2(\Delta t)}{\lambda^2}.
\end{aligned}
$$

It is immediate that

$$
P\left[\left|\int_{t_0}^t b_s ds\right| > \frac{\lambda}{4}\right] \leq \frac{4A(\Delta t)}{\lambda}.
$$

Now,

$$\begin{aligned}
P\left[\left|\int_{t_0}^{t}\int_{|z|>1} f\left(X_s,z\right)\mu\left(ds,dz\right)\right|>\frac{\lambda}{4}\right] &\le P\left[A\mu\left(\left[t_0,t\right]\times\left\{|z|>1\right\}\right)>\frac{\lambda}{4}\right]\\
&\le \frac{4}{\lambda}E\left[A\mu\left(\left[t_0,t\right]\times\left\{|z|>1\right\}\right)\right]\\
&\le \frac{4A\left(\Delta t\right)\nu\left\{|z|>1\right\}}{\lambda}
\end{aligned}$$

and

$$\begin{aligned}
P\left[\left|\int_{t_0}^{t}\int_{|z|\le 1} f\left(X_s,z\right)\widetilde{\mu}\left(ds,dz\right)\right|>\frac{\lambda}{4}\right] &\le \frac{16}{\lambda^2}E\left[\left|\int_{t_0}^{t}\int_{|z|\le 1} f\left(X_s,z\right)\widetilde{\mu}\left(ds,dz\right)\right|^2\right]\\
&\le \frac{16}{\lambda^2}\int_{t_0}^{t}ds\int_{|z|\le 1}E\left[\left|f\left(X_s,z\right)\right|^2\right]\nu\left(dz\right)\\
&\le \frac{16A^2\left(\Delta t\right)\nu\left\{|z|\le 1\right\}}{\lambda^2}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\Delta X_s\right|\le\varepsilon\left(\eta\right)\right\} \ge\ & 1-P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}b_u du\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\}\\
&-P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\sigma_u dW_u\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\}\\
&-P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|z|>1} f\left(X_{s-},z\right)\mu\left(ds,dz\right)\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\}\\
&-P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|z|\le 1} f\left(X_{s-},z\right)\widetilde{\mu}\left(ds,dz\right)\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\}
\end{aligned}$$

We easily get

$$P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\sigma_u dW_u\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\}\le\frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}.$$

$$P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}b_u du\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\}\le\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}$$

$$\begin{aligned}
& P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|z|>1} f\left(X_{s-},z\right)\mu\left(ds,dz\right)\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\} \\
\leq\; & P\left\{A\mu\left(\left[t_0,t_0+\Delta t\right]\times\left\{|z|>1\right\}\right)>\frac{\varepsilon\left(\eta\right)}{4}\right\} \\
\leq\; & \frac{4A}{\varepsilon\left(\eta\right)}E\left[\mu\left(\left[t_0,t_0+\Delta t\right]\times\left\{|z|>1\right\}\right)\right] \\
\leq\; & \frac{4A\left(\Delta t\right)\nu\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}
\end{aligned}$$

$$\begin{aligned}
& P\left\{\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|z|\leq 1} f\left(X_{s-},z\right)\widetilde{\mu}\left(ds,dz\right)\right|>\frac{\varepsilon\left(\eta\right)}{4}\right\} && (6)\\
\leq\; & \frac{16}{\varepsilon\left(\eta\right)^2}E\left[\sup_{s\in[t_0,t_0+\Delta t]}\left|\int_{t_0}^{s}\int_{|z|\leq 1} f\left(X_{s-},z\right)\widetilde{\mu}\left(ds,dz\right)\right|^2\right] && (7)\\
\leq\; & \frac{4\times 16}{\varepsilon\left(\eta\right)^2}E\left[\left|\int_{t_0}^{t_0+\Delta t}\int_{|z|\leq 1} f\left(X_{s-},z\right)\widetilde{\mu}\left(ds,dz\right)\right|^2\right] && (8)\\
\leq\; & \frac{64}{\varepsilon\left(\eta\right)^2}\int_{t_0}^{t_0+\Delta t} ds\int_{|z|\leq 1} E\left[\left|f\left(X_{s-},z\right)\right|^2\right]\nu\left(dz\right) \\
\leq\; & \frac{64A^2\left(\Delta t\right)\nu\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2}
\end{aligned}$$

where (8) is obtained by Doob's inequality and the fact that $\int_{t_0}^{s}\int_{|z|\leq 1} f\left(X_{s-},z\right)\widetilde{\mu}\left(ds,dz\right)$ is a martingale (Ikeda Watanabe, p62).

Then, as mentioned at the beginning of the subsection, if there exists $(\lambda,\Delta t,\eta,n)\in\left(\mathbb{R}_+\right)^3\times\mathbb{N}$ for which for all $\Delta t\leq t^*$ the condition

$$\begin{aligned}
& 2P\left(B_{t_0}\right)-\frac{32\eta^2\left(\Delta t\right)}{\lambda^2}-\frac{4A\left(\Delta t\right)}{\lambda}-\frac{4A\left(\Delta t\right)\nu\left\{|z|>1\right\}}{\lambda}-\frac{16A^2\left(\Delta t\right)\nu\left\{|z|\leq 1\right\}}{\lambda^2} \\
& -\frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}-\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}-\frac{4A\left(\Delta t\right)\nu\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}-\frac{64A^2\left(\Delta t\right)\nu\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2} \\
& +\frac{P\left(B^{\xi}\right)P\left(B_{t_0}\right)}{2\pi\Delta t}\int_{\substack{a_{11}^0x+a_{12}^0y\geq 2\lambda+\xi\\ a_{21}^0x+a_{22}^0y\leq -2\lambda-\xi\\ |x|\leq\frac{\lambda 2^n-1}{2},|y|\leq\frac{\lambda 2^n-1}{2}}} e^{-\frac{x^2+y^2}{2\Delta t}}dxdy \\
>\; & 2P\left(B_{t_0}\right) && (9)
\end{aligned}$$

holds, then our problem is solved. Inequality (9) is equivalent to

$$
\begin{aligned}
& \frac{P\left(B^{\xi}\right)}{2\pi\Delta t} \int_{\substack{a_{11}^0 x + a_{12}^0 y \geq 2\lambda+\xi \\ a_{21}^0 x + a_{22}^0 y \leq -2\lambda-\xi \\ |x| \leq \frac{\lambda 2^n - 1}{2}, |y| \leq \frac{\lambda 2^n - 1}{2}}} e^{-\frac{x^2+y^2}{2\Delta t}} dxdy \\
> \quad & \frac{32\eta^2 (\Delta t)}{\lambda^2} + \frac{4A(\Delta t)}{\lambda} + \frac{4A(\Delta t)\nu\{|z|>1\}}{\lambda} + \frac{16A^2(\Delta t)\nu\{|z|\leq 1\}}{\lambda^2} \\
& + \frac{128(\Delta t)A^2}{[\varepsilon(\eta)]^2} + \frac{16(\Delta t)^2 A^2}{[\varepsilon(\eta)]^2} + \frac{4A(\Delta t)\nu(\{|z|>1\})}{\varepsilon(\eta)} + \frac{64A^2(\Delta t)\nu(\{|z|\leq 1\})}{\varepsilon(\eta)^2}
\end{aligned} \tag{10}
$$

Letting $\xi = \lambda$, $u = \frac{x}{\sqrt{\Delta t}}$, $v = \frac{y}{\sqrt{\Delta t}}$ and $\mu = \frac{\lambda}{\sqrt{\Delta t}}$, the inequality is equivalent to

$$
L_1 \equiv \frac{P\left(B^{\lambda}\right)}{2\pi} \int_{\substack{M\binom{u}{v} \geq \binom{3\mu}{3\mu} \\ |u| \leq \frac{\lambda 2^n - 1}{2\sqrt{\Delta t}}, |v| \leq \frac{\lambda 2^n - 1}{2\sqrt{\Delta t}}}} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv > L_2
$$

for

$$
\begin{aligned}
L_2 \quad \equiv \quad & \frac{32\eta^2}{\mu^2} + \frac{4A\sqrt{(\Delta t)}}{\mu} + \frac{4A\sqrt{(\Delta t)}\nu\{|z|>1\}}{\mu} + \frac{16A^2\nu\{|z|\leq 1\}}{\mu^2} \\
& + \frac{128(\Delta t)A^2}{[\varepsilon(\eta)]^2} + \frac{16(\Delta t)^2 A^2}{[\varepsilon(\eta)]^2} + \frac{4A(\Delta t)\nu(\{|z|>1\})}{\varepsilon(\eta)} + \frac{64A^2(\Delta t)\nu(\{|z|\leq 1\})}{\varepsilon(\eta)^2}
\end{aligned}
$$

where $M \equiv \begin{pmatrix} a_{11}^0 & a_{12}^0 \\ -a_{21}^0 & -a_{22}^0 \end{pmatrix}$. As we have seen in Section 3.1, for $n \geq n_0$, we know that for $i,j = 1,2$, $a_{ij}^0 \in \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$ on B_{t_0} and for all $a \in \prod_{i,j=1}^2 \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$, $a_{11}a_{22} - a_{21}a_{12} \neq 0$. Then there exist real numbers $\bar{\gamma}_{ij}$'s for which, letting $\bar{M} \equiv \begin{pmatrix} \bar{\gamma}_{11} & \bar{\gamma}_{12} \\ \bar{\gamma}_{21} & \bar{\gamma}_{22} \end{pmatrix}$, $\bar{M}$ is invertible and

$$
\bar{M}\binom{u}{v} \geq \binom{3\mu}{3\mu} \Rightarrow \begin{pmatrix} a_{11} & a_{12} \\ -a_{21} & -a_{22} \end{pmatrix} \binom{u}{v} \geq \binom{3\mu}{3\mu}
$$

for all $a \in \prod_{i,j=1}^2 \left[\underline{\sigma}_{ij}, \overline{\sigma}_{ij}\right]$. Then,

$$
L_1 \geq \frac{P\left(B^{\lambda}\right)}{2\pi} \int_{\substack{\binom{u}{v} \in \bar{M}^{-1}\left([3\mu;+\infty[^2\right) \\ |u| \leq \frac{\lambda 2^n - 1}{2\sqrt{\Delta t}}, |v| \leq \frac{\lambda 2^n - 1}{2\sqrt{\Delta t}}}} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv
$$

Since $\bar{M}^{-1}\left([3\mu;+\infty[^2\right)$ is independent of n and since we can choose n as large as we want (greater than n_0), we only need to solve

$$
\frac{P\left(B^{\lambda}\right)}{2\pi} \int_{\binom{u}{v} \in \bar{M}^{-1}\left([3\mu;+\infty[^2\right)} e^{-\frac{1}{2}\left(u^2+v^2\right)} dudv > L_2.
$$

As for $P\left(B^{\lambda}\right)$, we have

$$\begin{aligned} P\left(B^{\lambda}\right) &= P\left\{\left|\Delta W_{t_0+\Delta t}^{j}\right| \leq \frac{\lambda}{2\left(d-2\right)A}, j=3,...,d\right\} \\ &\geq \prod_{j=3}^{d}\left\{1-\frac{4\left(d-2\right)^2A^2}{\lambda^2}E\left[\left(\Delta W_{t_0+\Delta t}^{j}\right)^2\right]\right\} \\ &\geq \left[1-\frac{4\left(d-2\right)^2A^2}{\lambda^2}\Delta t\right]^{d-2}. \end{aligned}$$

Let $\varphi\left(\mu\right) \equiv \frac{1}{2\pi}\int_{\binom{u}{v}\in\bar{M}^{-1}\left([3\mu;+\infty[^2\right)} e^{-\frac{1}{2}\left(u^2+v^2\right)}dudv$. We fix then μ such that

$$\frac{4A}{\mu}+\frac{4A\nu\left\{|z|>1\right\}}{\mu}+\frac{16A^2\nu\left\{|z|\leq 1\right\}}{\mu^2}<\frac{1}{6}\varphi\left(\mu\right)$$

and $\left[1-\frac{4(d-2)^2A^2}{\lambda^2}\Delta t\right]^{d-2}>\frac{1}{2}$, we find η such that $\frac{32\eta^2}{\mu^2}<\frac{1}{6}\varphi\left(\mu\right)$, then $(\Delta t)<1$ such that $\frac{128(\Delta t)A^2}{[\varepsilon(\eta)]^2}+\frac{16(\Delta t)^2A^2}{[\varepsilon(\eta)]^2}+\frac{4A(\Delta t)\nu(\{|z|>1\})}{\varepsilon(\eta)}+\frac{64A^2(\Delta t)\nu(\{|z|\leq 1\})}{\varepsilon(\eta)^2}<\frac{1}{6}\varphi\left(\mu\right)$ and $\lambda \equiv \mu\sqrt{\Delta t}$. This enables us to get

$$\begin{aligned} P\left(B^{\lambda}\right)\varphi\left(\mu\right) &> \frac{1}{2}\varphi\left(\mu\right) \\ &> \frac{32\eta^2}{\mu^2}+\frac{4A}{\mu}+\frac{4A\nu\left\{|z|>1\right\}}{\mu}+\frac{16A^2\nu\left\{|z|\leq 1\right\}}{\mu^2} \\ &\quad \frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{4A\left(\Delta t\right)\nu\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}+\frac{64A^2\left(\Delta t\right)\nu\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2}. \\ &> \frac{32\eta^2}{\mu^2}+\frac{4A\sqrt{(\Delta t)}}{\mu}+\frac{4A\sqrt{(\Delta t)}\nu\left\{|z|>1\right\}}{\mu}+\frac{16A^2\nu\left\{|z|\leq 1\right\}}{\mu^2} \\ &\quad \frac{128\left(\Delta t\right)A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{16\left(\Delta t\right)^2A^2}{\left[\varepsilon\left(\eta\right)\right]^2}+\frac{4A\left(\Delta t\right)\nu\left(\left\{|z|>1\right\}\right)}{\varepsilon\left(\eta\right)}+\frac{64A^2\left(\Delta t\right)\nu\left(\left\{|z|\leq 1\right\}\right)}{\varepsilon\left(\eta\right)^2}. \end{aligned}$$

We have then the existence of $\left(\lambda_1,\left(\Delta t\right)_1,\eta_1,n_1\right)\in\left(\mathbb{R}_+^*\right)^3\times\mathbb{N}$ such that Inequation (5) holds. Remark that we can take Δt as small as we want as long as we modify λ accordingly.

Proceding in the exact same way for the set $A_{\lambda,\Delta t,\eta,n}^2$, we get the existence of $\left(\lambda_2,\left(\Delta t\right)_2,\eta_2,n_2\right)\in\left(\mathbb{R}_+^*\right)^3\times\mathbb{N}$ such that $P\left[A_{\lambda_2,\left(\Delta t\right)_2,\eta_2,n_2}^2\right]>0$; now, taking $n=\sup\left(n_1,n_2\right)$ and $\Delta t=\inf\left[\left(\Delta t\right)_1,\left(\Delta t\right)_2\right]$, we obtain that there exist $\left[\left(\lambda_i,\eta_i\right)\right]_{i=1,2}\in(\mathbb{R}_+^*)^2\times(\mathbb{R}_+^*)^2$, $n\in\mathbb{N}$ and $\Delta t\in\mathbb{R}_+^*$ for which $P\left[A_{\lambda_i,\Delta t,\eta_i,n}^i\right]>0$ for $i=1,2$, which, using Lemma 1, completes the proof.

2.4 m-dimensional Processes

We now assume that the process X is an m-dimensional Markov process, solution of a stochastic differential equation with jumps, for m possibly greater than 2. As in the preceding subsection, let $W = \left\{ \left(W_t^1, ..., W_t^d \right)^* ; t \geq 0 \right\}$ denote a d -dimensional Brownian motion for $(F_t)_{t \geq 0}$. Let $\mathcal{M}^{m,d}$ denote the set of real valued $(m \times d)$ -matrices. Let $\sigma : \mathbb{R}^m \to \mathcal{M}^{m,d}$ and $b : \mathbb{R}^m \to \mathbb{R}^m$ and $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m$ be Borel measurable and uniformly continuous functions such that for some positive constants A and K, Inequations(2) and (3) are satisfied. Under these conditions, we know that the stochastic differential equation (4) with given initial condition $X_0 = \left(X_0^1, ..., X_0^m \right)$, where X_0 is supposed to be a square integrable R^m-valued F_0 -measurable random variable, admits a unique continuous, $(F_t)_{t \geq 0}$-adapted m -dimensional solution process $X = \left\{ \left(X^1, ..., X^m \right)^* \right\}$. We shall prove the following

Theorem 2 *If the real-valued solution processes X^1 and X^2 of Equation (4) are comonotonic, then for all t, their dispersion coefficients are linked by the following relation*

$$\sigma_{1j}(t)\,\sigma_{2j'}(t) - \sigma_{2j}(t)\,\sigma_{1j'}(t) = 0 \qquad P \text{ a.s. for all } 1 \leq j, j' \leq d.$$

Proof The proof is similar to the one made in the case $m = 2$. We consider the same specific set B_{t_0} at time t_0 and the same sets $B^1_{\lambda,\Delta t,\eta,n}$ and $B^2_{\lambda,\Delta t,\eta,n}$ for all $(\lambda, \Delta t, \eta, n) \in (\mathbb{R}_+)^3 \times \mathbb{N}$. Lemma 1 remains valid. Then, we show exactly like in the preceding section that there exist $(\lambda, \Delta t, \eta, n) \in \left(\mathbb{R}_+^* \right)^3 \times \mathbb{N}$ for which the condition of Lemma 1 holds.□

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