Production planning and inventories optimization: A backward approach in the convex storage cost case

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Abstract

We study the deterministic optimization problem of a profit-maximizing firm which plans its sales/production schedule. The firm controls both its production and sales rates and knows the revenue associated to a given level of sales, as well as its production and storage costs. The revenue and the production cost are assumed to be respectively concave and convex. In Chazal et al. [Chazal, M., Jouini, E., Tahraoui, R., 2003. Production planning and inventories optimization with a general storage cost function. Nonlinear Analysis 54, 1365–1395], we provide an existence result and derive some necessary conditions of optimality. Here, we further assume that the storage cost is convex. This allows us to relate the optimal planning problem to the study of a backward integro-differential equation, from which we obtain an explicit construction of the optimal plan.

JEL classification: D21; D42

Keywords: Production planning; Inventory management; Integro-differential equations

1. Introduction

We consider a firm which produces and sells a good which can be stored. The firm acts in continuous time on a finite period in order to maximize dynamically its profit. Its instantaneous profit is given by the revenue rate entailed by the sales, diminished by the production cost rate, and by the cost of storage of the current inventories. Our approach to this production planning and inventory management problem is in the same vein as the one launched by Arrow et al. (1958).

Many contributions to this theory have been brought from the 1950s until now, with different approaches. A considerable part of the literature is devoted to the study of cost-minimizing firms. Basically, in production-planning models, the firm makes decisions in order to minimize its production and storage costs and on the basis of the good demand forecast. In production-smoothing models, which were initiated by Holt et al. (1960), additional adjustment

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cost are taken into account. Some alternative to these models is given by the (S, s) models of inventory behavior. See notably Scarf (1960) for a basic existence result. A unified mathematical treatment of production-planning and production-smoothing problems was proposed by Bensoussan et al. (1983). Applications of optimal control theory in management science and in particular in production planning can be found in Sethi and Thompson (1981) or in Maimon et al. (1998), for more recent results.

All models of this prominent part of the production-planning literature assume that the sales of the firm are completely driven by the possibly time-varying and stochastic demand. In this paper, we work in a deterministic context: the firm knows the revenue associated to a certain level of sales as well as the cost of production and the cost of storage. In addition, we assume that the firm controls not only its production rate but also its sales rate. When the demand of the product is a bijective function of the price, this amounts to assume that the firm chooses the price. This is typically the case of a monopolistic firm. Our model is closely related to the one by Pekelman (1974), who was probably the first to take price as a decision variable. Feichtinger and Hartl (1985) extended Pekelman's work to the case of a general convex storage cost function (not necessarily linear) as well as to the case of a nonlinear demand function. Following these works, we will assume that the production and storage costs are given by some general convex functions. To the contrary, we will not impose the revenue function to be computed as the price times the demand function. It will be given by some general concave function of the sales rate. Another main difference with Feichtinger and Hartl's model is that we do not allow for backlogging.

The sales/production-planning problem of the profit-maximizing firm is formulated as an optimal control problem where the controls, namely the sales and production paths, are integrable. In other words, the cumulative production and sales processes are assumed to be absolutely continuous. The investigation of the problem is not straightforward due to the presence of control and state constraints. It has been studied by the same authors in Chazal et al. (2003) with a general storage cost function. We proved an existence result for a relaxed problem in which the cumulative sales process is allowed to have a jump at time 0 (i.e. the firm is allowed to make a partial depletion of its eventual initial inventory at time 0). We also derived the first order conditions of optimality for both problems and thus provided a qualitative description of the optimal plans. In particular, we establish the following result: the optimal inventory level must decrease until it reaches 0. It is then kept null by producing for immediate selling.

In the present article, we go further in this analysis. For this purpose, we assume that the storage cost is convex. In this context, we first see that the initial problem has a solution if and only if the relaxed one has a solution without jump at time 0. Second, the first order conditions of Chazal et al. (2003) allow us to characterize the (unique) optimal plan for the relaxed problem. Using this characterization, we relate the relaxed optimization problem to the study of a class of solutions of a backward integro-differential equation, indexed by a class of admissible terminal conditions. Using some standard existence result for retarded equations given in Hale and Verduyen Lunel (1993), we establish existence and uniqueness of a maximal solution for this backward integro-differential equation. We then study the behavior of the solution with respect to the terminal condition. This allows us to provide a constructive description of the optimal plan. The optimal plan is determined by selecting the greatest terminal condition r such that a certain functional of the solution of the backward integro-differential equation, representing the inventory level at time 0+, $S_0(r)$, remains lower than the exogenous initial inventory s_0. The difference $\alpha = s_0 - S_0(r)$ corresponds to the size of the jump of the cumulative sales process at time 0. After 0, the sales and production rates are explicitly determined in function of the corresponding maximal solution.

We obtain that there exists an exogenous threshold $\bar{s}_0$ on the initial inventory above which the size of the jump, α, is positive. The economic interpretation is the following. If the initial inventory s_0 entails too high storage costs, i.e. s_0 is greater than $\bar{s}_0$, then it is optimal for the firm to sell out immediately the quantity $s_0 - \bar{s}_0$, so as to reduce its initial inventory to $\bar{s}_0$. If the initial inventory is lower than $\bar{s}_0$, then the firm has no interest in selling out immediately. The level $\bar{s}_0$ appears as the maximal level that it can afford to hold. We also obtain that this threshold depends on the length of the planning period and we observe some qualitative differences between the optimal sales/production plans obtained on the short planning periods and on the long ones.

The paper is organized as follows. Section 2 provides a precise description of the model and recalls some basic results of Chazal et al. (2003). Section 3 is devoted to the backward characterization of the optimal plan for the relaxed problem. The constructive resolution of the production-planning problem is given in Section 4. The proofs of our main results are given in Section 5 by the way of the study of the above mentioned backward integro-differential equation.

2. The model formulation

The firm acts in continuous time on a finite period $[0, T]$. It is endowed with an initial inventory of $s_0 \in \mathbb{R}^+$ units of the good.

A sales/production plan is represented by a couple (x, y) of functions in $L^1_+[0, T]$, the set of nonnegative elements of $L^1[0, T]$, where $x(t)$ (respectively $y(t)$) is the sales (respectively production) rate in units of the good at time t. In other words $\int_0^t x(u)\mathrm{d}u$ (respectively $\int_0^t y(u)\mathrm{d}u$) is the cumulative quantity of the good sold out (respectively produced) up to time t. We shall say that $(x, y) \in L^1_+[0, T] \times L^1_+[0, T]$ is a sales/production plan if the induced inventory $S^{(x,y)}$ satisfies

$$S^{(x,y)}(t) \triangleq s_0 + \int_0^t y(u)\mathrm{d}u - \int_0^t x(u)\mathrm{d}u \geq 0, \quad \forall t \in [0, T].$$

This means that the company must never be out of stock. We denote by $\mathcal{A}$ the set of all sales/production plans:

$$\mathcal{A} \triangleq \{(x, y) \in L^1_+[0, T] \times L^1_+[0, T] | S^{(x,y)}(t) \geq 0, \quad \forall t \in [0, T]\}.$$

When selling out at the rate $x(t)$ at time t, the firm has a revenue rate of $\pi(x(t))$. The cost of producing at the rate $y(t)$ at time t is $c(y(t))$. Both π and c are continuous, non-decreasing functions on $\mathbb{R}^+$. They satisfy $\pi(0) = 0$, $c(0) = 0$ and $\pi(x) > 0$, $c(x) > 0$, for all positive x. The function π (respectively c) is assumed to be concave (respectively convex).

The cost of storing an amount $S(t)$ of goods at time t is denoted by $s(S(t))$. The function s is assumed to be continuous, non-decreasing on $\mathbb{R}^+$ and to satisfy $s(0) = 0$ and $s(S) > 0$, for all positive S.

Given the discount rate $\lambda > 0$, the profit over time induced by $(x, y) \in \mathcal{A}$ is defined by

$$J(x, y) \triangleq \int_0^T \mathrm{e}^{-\lambda t}[\pi(x(t)) - c(y(t)) - s(S^{(x,y)}(t))]\,\mathrm{d}t.$$

Observe that by concavity of π and Jensen's inequality we have $\int_0^T \mathrm{e}^{\lambda t}\pi(x(t))\,\mathrm{d}t < \infty$. The functions c and s being nonnegative, it follows that J is well defined as a map from $\mathcal{A}$ into $\mathbb{R} \cup \{-\infty\}$.

The profit-maximizing company plans its sales/production schedule by solving the following optimization problem:

$$\sup_{(x,y) \in \mathcal{A}} J(x, y). \tag{1}$$

It turns out that the function J may fail to have a maximum on $\mathcal{A}$. This is typically the case when s_0 is too high (see Section 4). Nevertheless, it was proved in Chazal et al. (2003) that existence holds for some relaxed problem that we now describe.

The sales rate is no longer described by an integrable function, but by a nonnegative finite Borel measure on $[0, T]$ which has its singular part positively proportional to the Dirac measure at 0. In this framework, for a sales path equal to $\alpha\delta_0 + x$, where $x \in L^1_+[0, T]$ represents the absolutely continuous part of the considered Borel measure, the cumulative sales process is given by

$$X(0) = 0 \quad \text{and} \quad X(t) = \alpha + \int_0^t x(u)\,\mathrm{d}u, \quad \forall t \in (0, T].$$

This means that the firm is allowed to sell out, at time 0, a share α of its initial inventory.

The production rate is still assumed to be integrable. A sales/production plan is now a triplet (α, x, y) in $\mathbb{R}^+ \times L^1_+[0, T] \times L^1_+[0, T]$ which satisfies the inventory constraint:

$$S^{(\alpha,x,y)}(t) = s_0 + \int_0^t y(u)\mathrm{d}u - \alpha - \int_0^t x(u)\mathrm{d}u \geq 0, \quad \forall t \in (0, T].$$

For $t = 0$, we set $S^{(\alpha,x,y)}(0) = s_0$. The inventory level $S^{(\alpha,x,y)}$ can have a downward jump ($\alpha \geq 0$) at $0+$.

We denote by $\mathcal{B}$ the set of relaxed sales/production plans:

$$\mathcal{B} \triangleq \{(\alpha, x, y) \in \mathbb{R}^+ \times L^1_+[0, T] \times L^1_+[0, T] | \;\; S^{(\alpha,x,y)}(t) \geq 0, \quad \forall t \in (0, T]\}.$$

The relaxed profit is defined on $\mathcal{B}$ by

$$\mathcal{F}(\alpha, x, y) = \alpha\dot{\pi}(\infty) + \int_0^T \mathrm{e}^{-\lambda t}[\pi(x(t)) - c(y(t)) - s(S^{(\alpha,x,y)}(t))]\,\mathrm{d}t$$

where we have set $\dot{\pi}(\infty) \triangleq \lim_{x\to\infty} \pi(x)/x$, which is well defined in $\mathbb{R}^+$, by concavity and nonnegativity of π. This is the price at which the firm can sell at an infinite rate. It is also the lowest price accessible for the company. However, since holding inventories has a cost, the firm may take advantage of an immediate depletion, even at this price. Observe that, when π is differentiable, we have $\lim_{x\to\infty} (\pi(x)/x) = \lim_{x\to\infty} \dot{\pi}(x)$.

The relaxed optimization problem is

$$\sup_{(\alpha,x,y)\in\mathcal{B}} \mathcal{F}(\alpha, x, y). \tag{2}$$

It was proved in Chazal et al. (2003) that

Theorem 1. *$\mathcal{F}$ has a maximum in $\mathcal{B}$ and* $\max_{(\alpha,x,y)\in\mathcal{B}} \mathcal{F}(\alpha, x, y) = \sup_{(x,y)\in\mathcal{A}} J(x, y)$.

Notice that in the context of Chazal et al. (2003), where the storage cost is not assumed to be convex, uniqueness is not guaranteed in Problem (2). In this article, we shall go further in the analysis of the set of solutions of both problems. For this purpose, we require some regularity conditions on π, c and s and we assume that s is convex. In particular, this will allow us to assert that Problem (2) has a unique solution and thus that Problem (1) has a (unique) solution if and only if the solution of Problem (2) is in $\{0\} \times \mathcal{A}$.

To be more precise, we shall work under the following conditions.

Standing assumption (*H*).

(i) Argmax $(\pi - c) = \{a\}$ for some $a > 0$.
(ii) π is differentiable on $(0, \infty)$ and $\dot{\pi}$ is continuous and one to one on $[a, \infty)$.
(iii) c is differentiable on $\mathbb{R}^+$ and $\dot{c}$ is continuous and one to one on $[0, a]$.
(iv) s is continuously differentiable on $\mathbb{R}^+$.
(v) s is convex.

Assumptions (i)–(iv) were already required in Chazal et al. (2003) in order to obtain a precise characterization of the set of plans which satisfy the first order conditions of optimality in Problem (2). The existence result has been obtained under a weaker assumption: the functions π, c and s are continuous, and $\pi - c$ admits a, possibly not unique, maximum.

Theorem 2. *Problem (2) has a unique solution (α, x, y). Problem (1) has a solution if and only if $\alpha = 0$. If $\alpha = 0$ then, (x, y) is the unique solution of (1).*

Proof. Existence for Problem (2) is stated in Theorem 1 and was proved in Chazal et al. (2003). Moreover, Theorem 7 in Chazal et al. (2003) states that, if (α, x, y) is a solution then $x \geq a \geq y$ a.e. Since π and c are respectively strictly concave on $[a, \infty)$ and strictly convex on $[0, a]$, and since s is convex, uniqueness in Problem (2) is straightforward. By noticing that $(0, x, y) \in \mathcal{B} \Leftrightarrow (x, y) \in \mathcal{A}$ and that $\mathcal{F}(0, \cdot)_{|\mathcal{A}} \equiv J$ and recalling that by Theorem 1 $\max_{(\alpha,x,y)\in\mathcal{B}} \mathcal{F}(\alpha, x, y) = \sup_{(x,y)\in\mathcal{A}} J(x, y)$ the proof of Theorem 2 is easy to complete. $\square$

In light of Theorem 2, we see that planning the optimal sales/production schedule can be done by solving the relaxed problem. Indeed, if Problem (1) has a solution (x, y), then $(\alpha = 0, x, y)$ is the solution of the relaxed problem (2). Since, we further exhibit situations where the solution of Problem (2) is not regular, i.e. $\alpha > 0$ (see Section 4), meaning that Problem (1) has no solution, we must in actual fact consider Problem (2) to solve the planning problem. We will see that for some revenue and costs functions π, c and s and some initial inventory s_0, the firm cannot act in an optimal way without getting rid at time 0 of a certain share of s_0. From now on, we focus our attention on the relaxed problem and what we call the optimal plan is the unique solution (α, x, y) of Problem (2).

3. Characterization of the optimal plan

In this section we begin with recalling some characterization and thus some qualitative description of the optimal plan derived from the first order conditions obtained in Chazal et al. (2003). We then introduce some backward formulation of these optimality conditions in order to reach our main goal here: to provide a constructive resolution of the planning problem.

3.1. The first order conditions

It was shown in Chazal et al. (2003) that the optimal plan must be such that there is no inventory accumulation. In particular, if the firm has no starting inventories ($s_0 = 0$) then it adopts a static strategy consisting in producing for immediate sales. Moreover, the concavity of the revenue and the convexity of the cost of production imposes to the firm to minimize the variations of its sales and production rates, so that it must produce and sell at the same constant rate a which maximizes $\pi - c$. To sum up, when $s_0 = 0$, the optimal plan is $(0, a, a)$. Let us now turn to the case where s_0 is positive. Remark that it is worth studying this case as time 0 can be seen as the current time and not necessarily the date where the firm starts its activity. Moreover, in our framework the firm has a control on its initial inventory and we prove that it is not optimal for the firm to get rid of its initial inventory at time 0 (see condition (II) of Proposition 3 which implies that $\alpha < s_0$).

Standing assumption. The initial inventory s_0 is positive.

As a direct consequence of Theorem 12 in Chazal et al. (2003), we have the following characterization of the solution of Problem (2).

Proposition 3. *$(\alpha, x, y) \in \mathcal{B}$ is the solution of Problem (2) if and only if it satisfies:*

(I) *x (respectively y) is non-increasing (respectively non-decreasing) and $x \geq a \geq y$ on $[0, T]$,*
(II) *$S^{(\alpha,x,y)}(T) = 0$ and $T_0 \triangleq \inf\{t \in (0, T] | S^{(\alpha,x,y)}(t) = 0\} > 0$,*
(III) *if $T_0 < T$ then,*
  (a) *$S^{(\alpha,x,y)} = 0$ on $[T_0, T]$,*
  (b) *$x = y = a$ on $(T_0, T]$,*
  (c) *$x(T_0-) = a$.*
(IV) *$S^{(\alpha,x,y)}$ is decreasing on $(0, T_0]$,*
(V) *x and y are both continuous on $(0, T_0)$ and satisfy*

$$e^{-\lambda t} \dot{\pi}(x(t)) = \dot{\pi}(x(0+)) + \int_0^t e^{-\lambda u} \dot{s}\left(S^{(\alpha,x,y)}(u)\right) du, \quad \forall t \in (0, T_0) \tag{3}$$

$$y(t) = g(x(t)), \quad \forall t \in (0, T_0) \tag{4}$$

where

$$g(r) \triangleq \begin{cases} \dot{c}^{-1}(\dot{\pi}(r)) & if \ \ \dot{\pi}(r) > \dot{c}(0) \\ 0 & elsewhere \end{cases}, \quad \forall \ r \in [a, \infty) \tag{5}$$

(VI) *If $\alpha > 0$ then $x(0+) = \infty$.*

Proof. By Theorem 7 in Chazal et al. (2003), items (I)–(VI) are equivalent to

$$\limsup_{\substack{\varepsilon > 0 \\ \varepsilon \to 0}} \left\{ \frac{\mathcal{F}((\alpha, x, y) + \varepsilon(\beta, h, k)) - \mathcal{F}(\alpha, x, y)}{\varepsilon} \right\} \leq 0,$$

for all $(\beta, h, k) \in \mathbb{R} \times L^1[0, T] \times L^1[0, T]$ such that $(\alpha, x, y) + \varepsilon_0(\beta, h, k) \in \mathcal{B}$ for some $\varepsilon_0 > 0$. Then, Proposition 3 holds by concavity of $\mathcal{F}$ and uniqueness in Problem (2). □

Notice that Proposition 3 was obtained in Chazal et al. (2003) in the general case, i.e. for a general storage cost function. The profit functional was therefore non-convex and small variations techniques were used to obtain Eqs. (3) and (4). It turns out that, in the convex case, these necessary conditions of the calculus of variations (Euler equations) can be derived from the standard maximum principle for state constrained problems.

Proposition 3 furnishes some qualitative properties of the optimal plan in the general case. The main property is that the inventory level is non-increasing and must be null at the end of the period. Indeed, any strategy leading to a positive inventory at the terminal time can be improved by increasing the sales rate without changing the production. The fact that the optimal inventory level is non-increasing is less obvious. It can be seen, from the proof of Proposition 3 in Chazal et al. (2003), that it arises from the concavity of the revenue, the convexity of the production cost and the fact that holding inventories has a cost. Besides, the present-time preference for money that has the firm, which is described by the positive discount rate, also plays a role: the firm has interest in selling earlier, delaying its production. Thus the optimal sales (respectively production) rate is non-increasing (respectively non-decreasing).

Let us now put in light the qualitative description provided by Proposition 3. The optimal way to deplete the initial inventory is in two phases. This leads to a three phases sales/production plan. The first phase is devoted to the selling activity. The firm begins with possibly depleting a share $\alpha \geq 0$ of its initial inventory at time 0, selling at an infinite rate. The sales rate is then non-increasing or equivalently the marginal revenue $t \to \dot{\pi}(x(t))$ is non-decreasing. If the marginal revenue overtakes the lowest marginal cost of production $\dot{c}(0)$ then, the production activity actually starts (see (4) and (5)). During this second phase, the sales rate and the production rate are such that the marginal revenue and the marginal production cost remain equal: $\dot{c}(y(t)) = \dot{\pi}(x(t))$. The production rate is non-decreasing. During this destocking stage the sales rate is greater than a and the production rate lower. If inventories are all cleared before the end of the period ($T_0 < T$) then, the third phase starts: production and sales are at the same constant rate, a, maximizing the instantaneous profit $\pi - c$.

Remark that by (II), even if the firm is allowed to get rid of its whole initial inventory at time 0, it does not. A basic and intuitive reason for this is that α is sold out at time 0 at an infinite rate and hence at the lowest price $\dot{\pi}(\infty)$. However, we can go further in the analysis of the trade-off between immediately get rid of the initial inventory or not. For this purpose, we introduce a notation which will be useful to express the optimal depletion rate of the inventory, i.e. the difference between the optimal sales rate and the optimal production rate. We shall put:

$$\delta(r) \triangleq r - g(r), \quad \forall \; r \in [a, \infty).$$

The following remark embodies some useful properties of g and δ.

Remark 4.

(1) Under assumption (H) and by construction the function g is continuous and decreasing on $[a, \infty)$. Moreover, it satisfies $g(a) = \dot{c}^{-1}(\dot{\pi}(a)) = a$ and takes values in $[0, a]$.
(2) The function δ is continuous and increasing on $[a, \infty)$. It satisfies $\delta(a) = 0$ and hence it is positive on (a, ∞). Moreover $\lim_{r\to\infty}\delta(r) = \infty$.

We can now give a reason why the firm keeps a positive level of initial inventory ($\alpha < s_0$): For a given depletion rate of the inventory level δ, the maximum of the instantaneous profit $\pi(x) - c(y)$ is obtained when $y = g(x)$ and $x - y = \delta$. Then, one can see that x and y are completely determined by the optimal depletion rate of the inventory and that the instantaneous profit is increasing with respect to δ. Now, if the share α of the initial inventory that is depleted at time 0 is increased then the storage cost is decreased but the depletion rate δ that the firm can start with is smaller (because of the positivity constraint on the inventory level) and thus the instantaneous profit is decreased.

3.2. The backward characterization

We now turn to our first step on the way to our constructive resolution of the planning problem. This first step consists in providing a backward procedure to find the optimal plan (α, x, y). This is Proposition 5 which states some sufficient conditions of optimality which are expressed by means of some integro-differential backward equation that must be satisfied by the (possibly translated) sales rate x.

Notation *For any function* f defined, nonnegative and continuous on some interval $(t_0, t_1]$, the limit $\lim_{\substack{t \to t_0 \\ t > t_0}} \int_t^{t_1} f(u)\mathrm{d}u$ exists in $[0, \infty]$ and we will denote it by $\int_{t_0}^{t_1} f(u)\,\mathrm{d}u$.

We can now state our backward procedure to find the optimal plan.

Proposition 5. *Consider some function* w and some couple $(r, \tau_0) \in [a, \infty) \times [0, T)$ *such that*:

(i) $w(T) = r$, *w is continuous and decreasing on* $(\tau_0, T]$,

(ii)
$$\textit{either}\ \ (r, \tau_0) \in (a, \infty) \times \{0\}, \quad \textit{or}\ \ (r, \tau_0) \in \{a\} \times [0, T), \tag{6}$$

$$\mathrm{e}^{-\lambda t}\dot{\pi}(w(t)) = \mathrm{e}^{-\lambda T}\dot{\pi}(r) - \int_t^T \mathrm{e}^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(v))\mathrm{d}v\right)\mathrm{d}u, \quad \forall t \in (\tau_0, T] \tag{7}$$

$$\textit{either}\ \int_{\tau_0}^T \delta(w(u))\mathrm{d}u = s_0, \quad \textit{or}\ \int_{\tau_0}^T \delta(w(u))\mathrm{d}u < s_0 \ \ \textit{and}\ \ w(\tau_0+) = \infty. \tag{8}$$

Then the optimal plan is given by:

$$\begin{pmatrix} \alpha \\ x \\ y \end{pmatrix} = \begin{pmatrix} s_0 - \displaystyle\int_{\tau_0}^T \delta(w(u))\mathrm{d}u \\ w(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]} \\ g\left(w(\cdot + \tau_0)\mathbf{1}_{(0, T-\tau_0]} + a\mathbf{1}_{(T-\tau_0, T]}\right) \end{pmatrix} \tag{9}$$

Proof. See Appendices A and B. □

First, Proposition 5 tells us that the optimal production rate can be obtained as an exogenous function of the optimal sales rate: we have $y = g(x)$ on the whole planning period. Therefore, in order to find the optimal plan, it suffices to determine the optimal sales policy (α, x). Second, we see that the optimal sales rate x will be obtained in terms of some solution of Eq. (7), characterized by the boundary conditions (6) and (8).

In the next section, by using the study of the set of solutions of the integro-differential equation involved in Proposition 5:

$$BW(r): \ \ \mathrm{e}^{-\lambda t}\dot{\pi}(w(t)) = \mathrm{e}^{-\lambda T}\dot{\pi}(r) - \int_t^T \mathrm{e}^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(v))\mathrm{d}v\right)\mathrm{d}u, \quad t \le T$$

when r varies in $[a, \infty)$, we will achieve our main aim to provide a constructive resolution of the production-planning problem.

4. Constructive resolution of the planning problem

In addition to the explicit determination of the optimal plan, we will give necessary and sufficient conditions for this optimal plan to be *regular* ($\alpha = 0$), i.e. without depletion at time 0. Given the revenue and costs functions and the length of the planning period, the regularity of the optimal plan depends only on the level of the initial inventory s_0. More precisely, it appears that there is a positive (possibly infinite) level $\bar{s}_0$, depending on π, c, s and T such that: if $s_0 \le \bar{s}_0$ (respectively $s_0 > \bar{s}_0$) then the solution is (not) regular. We will also exhibit some qualitative distinctions between the optimal schedule obtained on a long planning period or on a short one. All proofs are reported to Section 5. Theorems 8 and 10, which provide the optimal plan, are illustrated on some example.

We start with the existence and uniqueness of a maximal solution for $BW(r)$, given any terminal condition $r \in [a, \infty)$. From this maximal feature we deduce a necessary and sufficient condition for the solution to explode at some given time. This property may be needed when applying Proposition 5 in order to determine the optimal plan (see (8)). We also check that the solution has the continuity and monotonicity properties required by Proposition 5. For this purpose we work under the following technical assumptions.

Standing assumption.

- $(H\pi)$ The inverse of $\dot{\pi}_{|[a,\infty)}$ is locally Lipschtizian on $(\dot{\pi}(\infty), \dot{\pi}(a)]$.
- (Hc) The inverse of $\dot{c}_{|[0,a]}$ is locally Lipschitzian on $[\dot{c}(0), \dot{c}(a)]$.
- (Hs) The function s is strictly convex and $\dot{s}$ is locally Lipschitzian on $\mathbb{R}^+$.

Theorem 6. *For every $r \in [a, \infty)$, there exists a unique couple $(\tau(r), w)$, where $\tau(r) \in [-\infty, T)$ and w is a continuous function on $(\tau(r), T]$, such that:*

$$e^{-\lambda t}\dot{\pi}(w(t)) = e^{-\lambda T}\dot{\pi}(r) - \int_t^T e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(v))dv\right) du, \quad \forall t \in (\tau(r), T],$$

and

$$\text{either } \tau(r) = -\infty, \text{ or } \tau(r) > -\infty \text{ and } w(\tau(r)+) = \infty. \tag{10}$$

In addition, w is decreasing and takes values in $[a, \infty)$ and $\delta \circ w$ is continuous, decreasing and nonnegative on $(\tau(r), T]$.

Notation *Given $r \in [a, \infty)$,* if *$(\tau(r), w)$ is associated to r by* Theorem 6 *we will write in short w^r and say that w^r is the maximal solution of $BW(r)$.*

We denote by τ the map from $[a, \infty)$ into $[-\infty, T)$ defined by Theorem 6.

By using changes of variable, it is easy to deduce from Theorem 6 the following time homogeneity property for the equation $BW(r)$ (invariance by translation on the terminal time).

Remark 7.

(1) Let $r \in [a, \infty)$ and $T_1, T_2 \in [0, \infty)$. If $(w_1, \tau_1(r))$ (respectively $(w_2, \tau_2(r))$) is the maximal solution of $BW(r)$ with terminal time $T = T_1$ (respectively T_2) then we have: $T_1 - \tau_1(r) = T_2 - \tau_2(r)$ and $w_1(t) = w_2(t + T_2 - T_1)$, for any $t \in (\tau_1(r), T_1]$.
(2) It follows that the length of the domain of the maximal solution of $BW(a)$, $T_a \triangleq T - \tau(a)$, and the integral $s^a \triangleq \int_{\tau(a)}^T \delta(w^a(u))du$ depend only on π, c and s and not on the length of the planning period T.

In the sequel, we will say that the planning period is *long* (respectively *short*) if $T \in (T_a, \infty)$ (respectively if $T \in (0, T_a]$)).

Example. Let us now describe the example on which our main theorems are illustrated. We define π, c and s by setting $\pi(x) = 2(1 - e^{-x})$ and $c(x) = x^2/2 + x/2$ for all $x \in \mathbb{R}^+$ and $s(S) = S$ for all $S \in \mathbb{R}^+$. Let us set $\lambda = 0.5$. Notice that assumptions $(H\pi)$ and (Hc) are satisfied but (Hs) is not. However, it is easy to see that, in this framework, Theorems 6–10 still hold and after computations, we have: for every $r \in [a, \infty)$,

$$T_r \triangleq T - \tau(r) = 2\ln(e^{-r} + 1),$$

$$w^r(t) = -\ln(e^{(t-T)/2}(e^{-r} + 1) - 1), \quad \forall t \in (\tau(r), T].$$

We obtain
$a = 0.59886$, $T_a = 0.87578$ and $s^a = \int_{\tau(a)}^T [w^a(t) - g(w^a(t))]dt = 1.39717$.

4.1. The long planning period case

Theorem 8. *Assume that $T \in (T_a, \infty)$. Then $s^a \in (0, \infty]$ and we have*:

1. *If $s_0 \le s^a$ then there exists a unique $\tau_0 \in [0, T)$ such that $\int_{\tau_0}^{T} \delta(w^a(u))\mathrm{d}u = s_0$ and the optimal plan is regular, given by* (9) *with $w = w^a$ and the above τ_0.*
2. *If s^a is finite and $s_0 > s^a$ then the optimal plan is not regular, given by* (9) *with $w = w^a$ and $\tau_0 = \tau(a)$.*

Figs. 1 and 2 illustrate Theorem 8. We have set the horizon time to $T = 1.5$ so that $T \in (T_a, \infty)$. In Fig. 1, we see that if the initial inventory s_0 is lower than the maximal level that can be backward cumulated with w^a, i.e. s^a, then the optimal inventory level is obtained by translation from τ_0 to 0 of the backward cumulative inventory associated to w^a. Once it has reached 0, it remains equal to 0. We see that the optimal plan is regular, i.e. without depletion at time 0 and it is actually is three phases. In Fig. 2, we see that when the initial inventory is above the level s^a then the optimal inventory level jumps down to s^a at time 0. It is then given by the translation from $\tau(a)$ to 0 of the backward

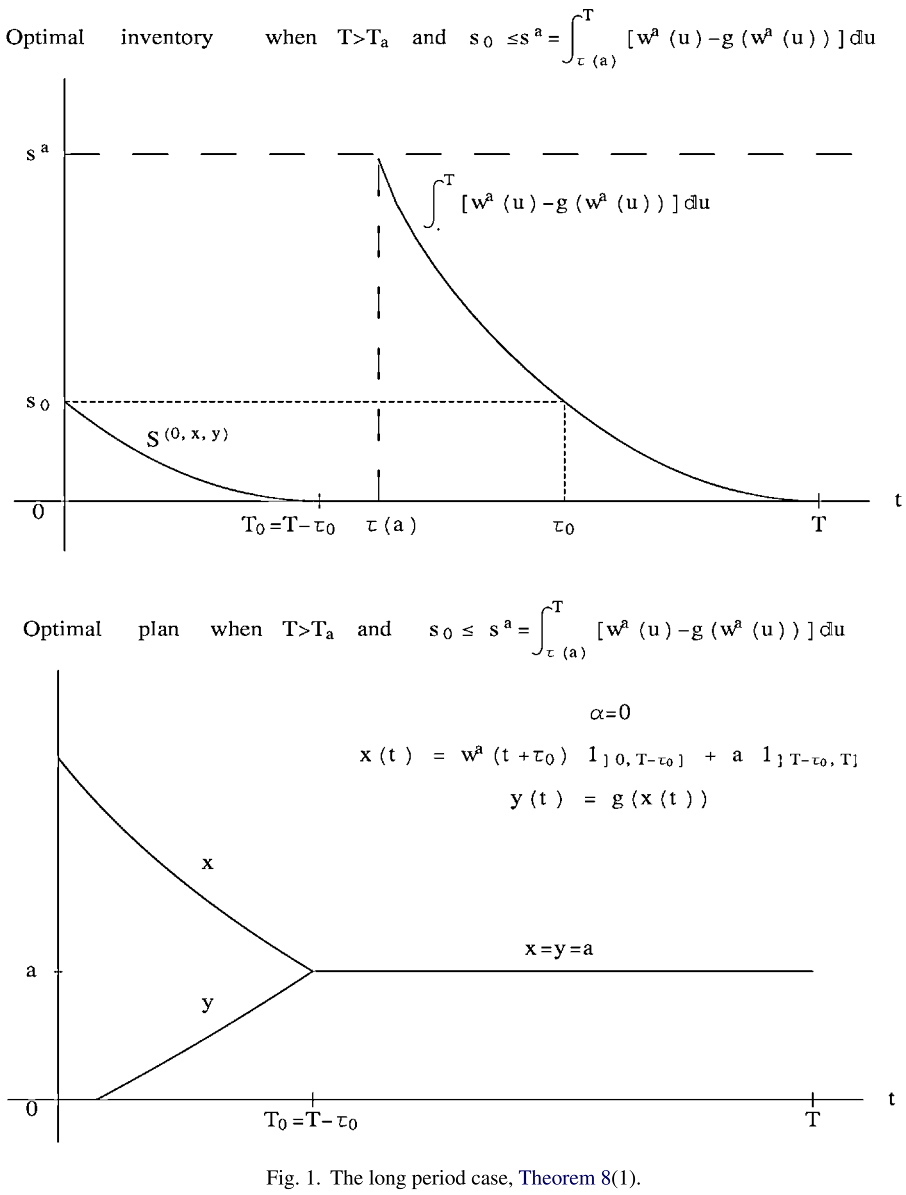


Fig. 1. The long period case, Theorem 8(1).

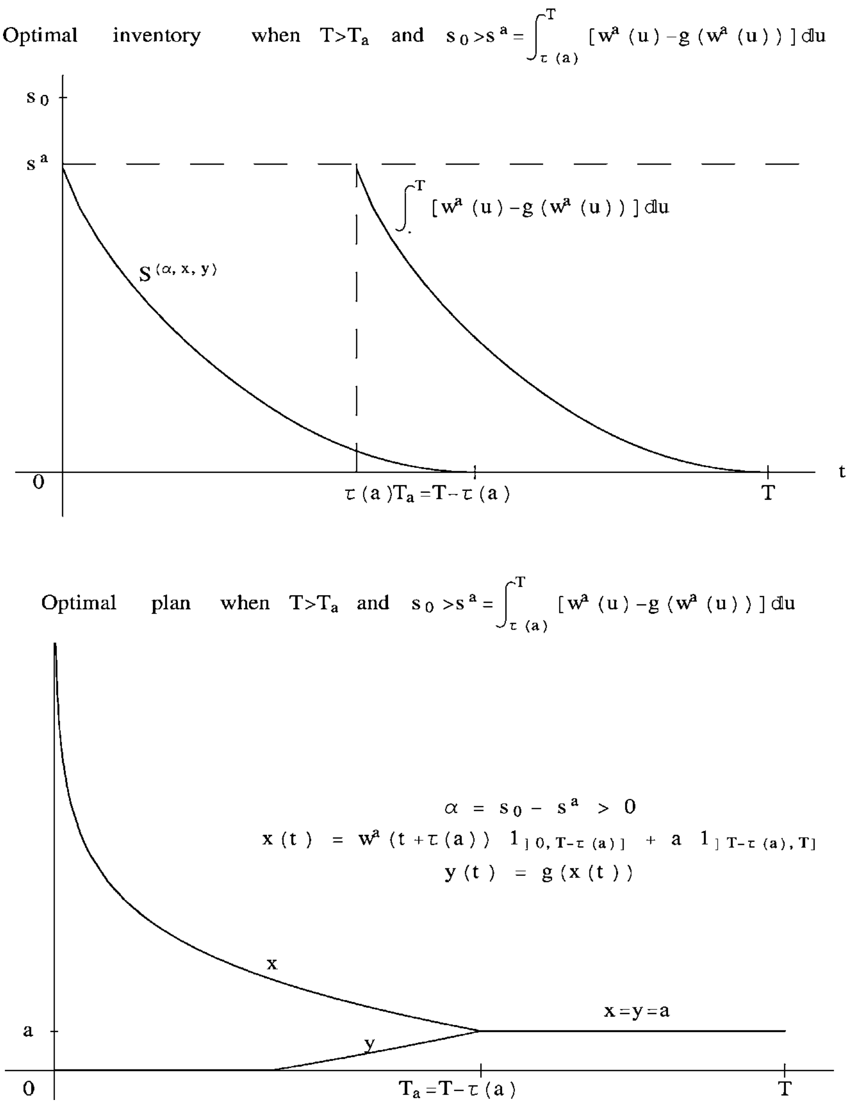


Fig. 2. The long period case, Theorem 8(2).

cumulative inventory associated to w^a. The sales start at an infinite rate.

From Theorem 8, we see that when the planning period is long ($T \in (T_a, \infty)$), the optimal plan is regular if and only if the initial inventory is below the exogenous level $\bar{s}_0 = s^a$. Notice that it may be infinite. In that case, whatever the initial inventory is, the optimal plan is regular.

From an economic point of view, $\bar{s}_0$ acts as a threshold. If the initial inventory s_0 is greater than $\bar{s}_0$, then holding such a (high) amount of stock is too expensive. It is then optimal for the firm to sell out immediately the quantity $\alpha = s_0 - \bar{s}_0$, so as to reduce its initial inventory to $\bar{s}_0$. This is the maximal level that it can afford to hold. However, we already mentioned (see the discussion following Remark 4) that the firm has no interest in selling more than necessary. This explains why it sells exactly $s_0 - \bar{s}_0$.

For every $s_0 > \bar{s}_0$, the regular part (x, y) of the optimal plan is the same, it corresponds to the optimal plan obtained for an initial inventory level equal to $\bar{s}_0$. Only the size of the jump on the sales $\alpha = s_0 - \bar{s}_0$ changes with s_0. If $s_0 \le \bar{s}_0$ then, the optimal plan is regular and actually depends on s_0 since it is obtained from the translation of w^a on the interval $(0, T - \tau_0]$ where τ_0 solves $s_0 = \int_{\tau_0}^{T} \delta(w^a(u))\mathrm{d}u$.

Remark that the existence of such a threshold and the corresponding properties of the optimal plan will also hold in the case of a short planning period. We will determine this threshold but not reproduce the discussion.

To the contrary what follows is specific to the long planning periods. First, the threshold $\bar{s}_0$ does not depend on T. Second, the plan that the firm adopts is in two (non-trivial) phases. The first phase consists in selling out the initial inventory in an optimal way. During the second phase, the firm follows the just-in-time strategy, producing and selling at the same constant rate a, until the end of the period. The depletion phase does not depend on T. It is the one associated to the optimal plan obtained on the period $[0, T_a]$. In view of item 2, this property is straightforward when $s_0 > \bar{s}_0 = \int_{\tau(a)}^{T} \delta(w^a(u))\mathrm{d}u$. When $s_0 \leq \bar{s}_0$, it follows from the time homogeneity property of $BW(a)$ (see Remark 7).

We end up this first step of the constructive resolution by stressing the fact that, in the case of a long planning period, nothing more than the computation of w^a is necessary to solve the planning problem. We can now turn to the short planning period case.

4.2. The short planning period case

Theorem 9. *Assume that $T \in (0, T_a]$. Then we have $\int_0^T \delta(w^a(u))\mathrm{d}u \in (0, \infty]$ and if $s_0 \leq \int_0^T \delta(w^a(u))\mathrm{d}u$ then there exists a unique $\tau_0 \in [0, T)$ such that $\int_{\tau_0}^T \delta(w^a(u))\mathrm{d}u = s_0$. The optimal plan is regular given by (9) with $w = w^a$ and the above τ_0.*

We see that even if the planning period is short, when the initial inventory s_0 is lower than $\int_0^T \delta(w^a(u))\mathrm{d}u$, the firm has time to clear out its stock before the horizon T, selling out at the rate $x = w^a(\cdot + \tau_0)$ on $(0, T - \tau_0]$ and it can afford a static management final phase: $x = y = a$ on $(\tau_0, T]$. We will find out that this is not the case when the initial inventory is greater than $\int_0^T \delta(w^a(u))\mathrm{d}u$.

Notice that if $T = T_a$, i.e. $\tau(a) = 0$, the quantity $\int_0^T \delta(w^a(u))\mathrm{d}u$ may be infinite and in that case, Theorem 9 put an end to the constructive resolution of the planning problem. From now on we assume that

$$\int_0^T \delta(w^a(u))\mathrm{d}u < \infty.$$

When $s_0 > \int_0^T \delta(w^a(u))\mathrm{d}u$, we will prove that the optimal sales rate is given by the solution $w^{\hat{r}}$ of $BW(\hat{r})$, for some $\hat{r} > a$, on the whole planning period. The difference between the optimal sales rate and the optimal production rate which is given by $\delta \circ w^{\hat{r}}$ will therefore be positive (see Theorem 6) on the whole planning period.

The terminal condition $\hat{r}$ characterizing the optimal sales rate will be the greatest one for which the domain of $w^{\hat{r}}$ is at least $(0, T]$ and such that the induced backward cumulative inventory level at time $0+$, $\int_0^T \delta(w^{\hat{r}}(u))\mathrm{d}u$, remains lower or equal to s_0. To see this we investigate the monotonicity and continuity with respect to r of the following map:

$$S_0 : \{r \in [a, \infty) | \tau(r) \leq 0\} \to \mathbb{R}^+ \cup \{\infty\}, \quad r \mapsto \int_0^T \delta(w^r(u))\mathrm{d}u.$$

In order to prove the existence of a threshold on the initial inventory below (above) which the optimal plan is (not) regular and in order to determine this threshold we also investigate the boundness of S_0: we seek for the greatest (finite) value that it can reach when the terminal condition r varies in $\{r \in [a, \infty) | \tau(r) \leq 0\}$. It appears that there are two cases to keep distinct. Theorem 10 below provides the optimal plan in each of these two cases. The study of the map S_0 can be read in detail in Section 5.3. Roughly speaking, we obtain that S_0 increases continuously with respect to r until it reaches some maximal (possibly infinite) level associated to a certain terminal condition. Above this terminal condition, S_0 becomes infinite or is no longer defined.

Theorem 10. *Assume that $T \in (0, T_a]$ and that $\int_0^T \delta(w^a(u))\mathrm{d}u < \infty$. Then, the set*

$$D_0 \triangleq \{r \in [a, \infty) | \tau(r) \leq 0 \text{ and } \int_0^T \delta(w^r(u))\mathrm{d}u < \infty\}$$

is a bounded interval with minimum equal to a. Let us denote $d \triangleq \sup D_0$ and assume that $s_0 > \int_0^T \delta(w^a(u))\mathrm{d}u$.

(1) *If* $\sup_{r \in D_0} \int_0^T \delta(w^r(u))\mathrm{d}u = \infty$ *then there exists a unique* $\hat{r} \in (a, d)$ *such that* $\int_0^T \delta(w^{\hat{r}}(u))\mathrm{d}u = s_0$ *and the optimal plan is regular, given by* (9) *with* $w = w^{\hat{r}}$ *and* $\tau_0 = 0$.

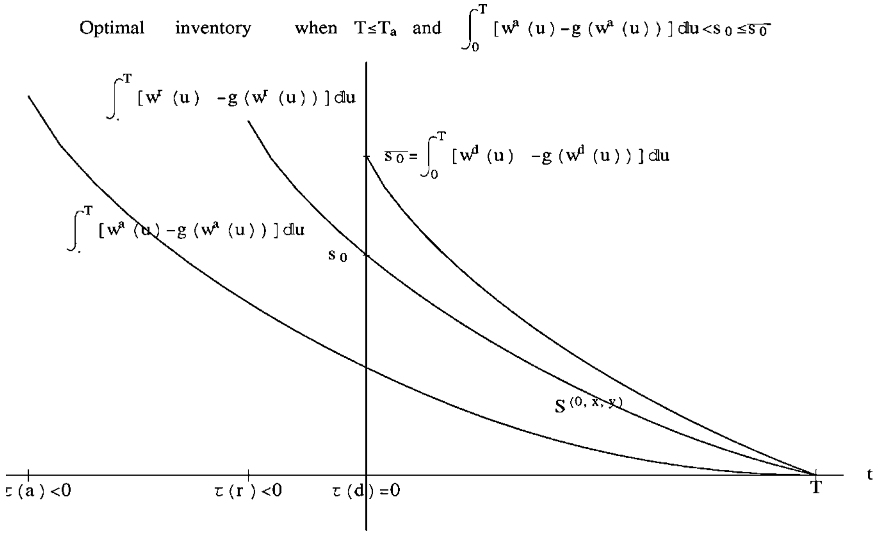
(2) *If* $\sup_{r \in D_0} \int_0^T \delta(w^r(u))\mathrm{d}u < \infty$ *then* $\tau(d) = 0$, $\int_0^T \delta(w^d(u))\mathrm{d}u < \infty$ *and*

  (a) *If* $s_0 \le \int_0^T \delta(w^d(u))\mathrm{d}u$ *then there exists a unique* $\hat{r} \in (a, d]$ *such that* $\int_0^T \delta(w^{\hat{r}}(u))\mathrm{d}u = s_0$ *and the optimal plan is regular, given by* (9) *with* $w = w^{\hat{r}}$ *and* $\tau_0 = 0$.

  (b) *If* $s_0 > \int_0^T \delta(w^d(u))\mathrm{d}u$ *then the optimal plan is not regular, given by* (9) *with* $w = w^d$ *and* $\tau_0 = 0$.

Figs. 3 and 4 illustrate Theorem 10. The horizon time is $T = 0.5$. For this example, we see that item 2 of Theorem 10 holds. Indeed, we have:

$$\bar{s}_0 \triangleq \sup_{r \in D_0} \int_0^T \delta(w^r(u))\mathrm{d}u = 1.15964 < \infty.$$



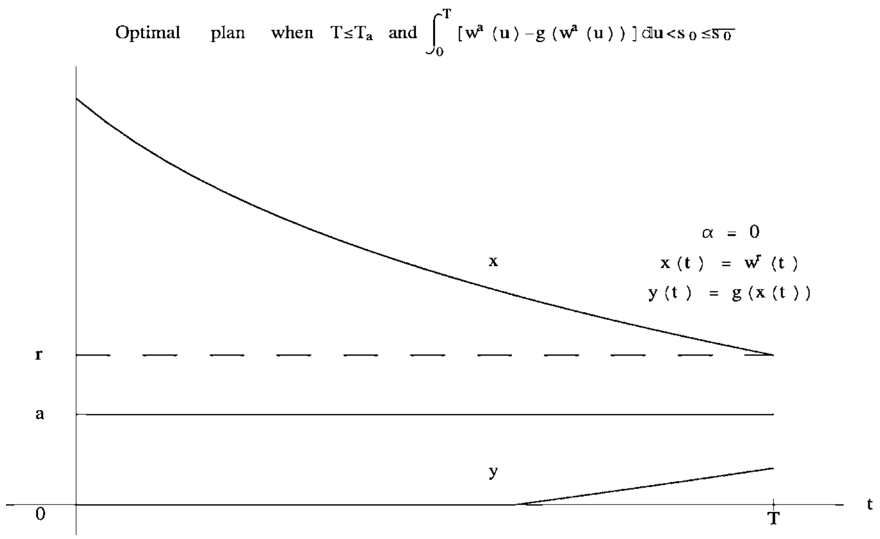


Fig. 3. The short period case, Theorem 10(2a).

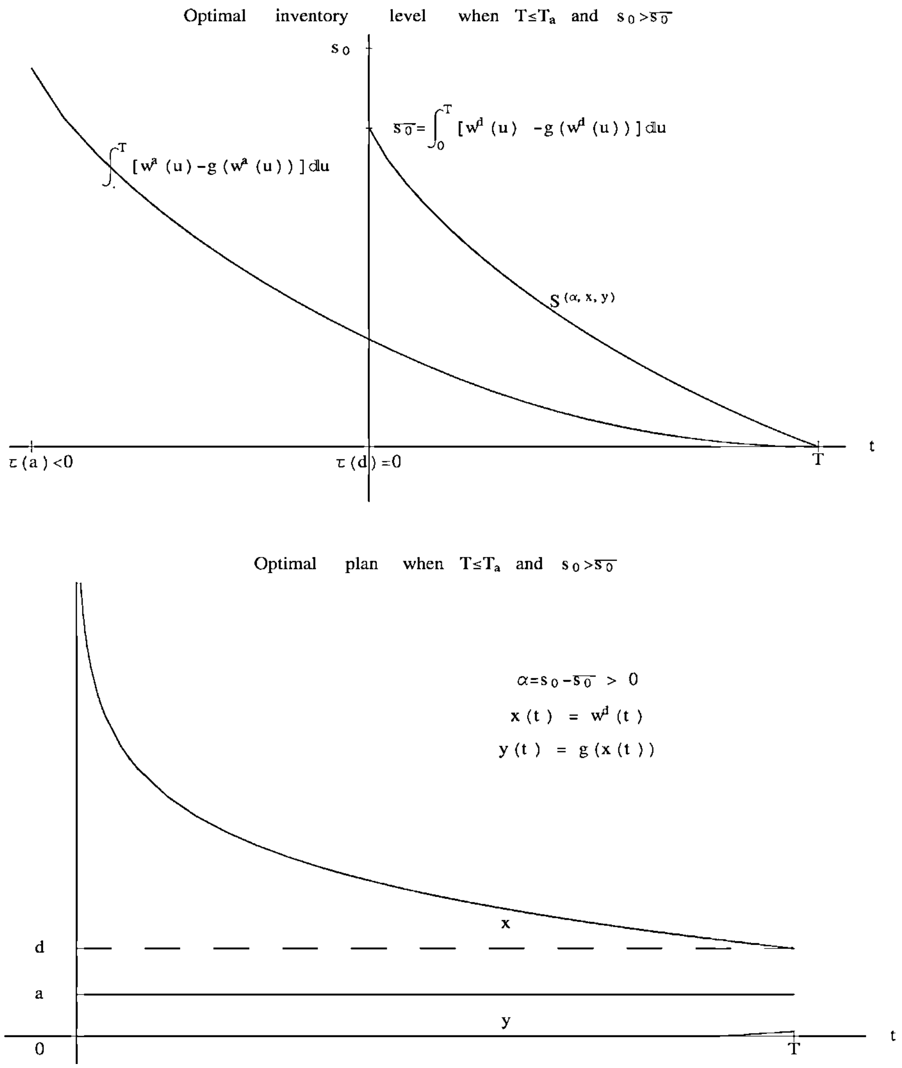


Fig. 4. The short period case, Theorem 10(2b).

This maximal level is the backward cumulative inventory level at time 0 associated to w^d where $d = 1.25869$. In Fig. 3 we have plotted the optimal inventory level for $s_0 = 0.8$. In that case, we have $s_0 < \bar{s}_0$, the optimal plan is therefore regular, i.e. without depletion at time 0. The optimal sales rate is given by w^r for $r = 0.99135$. We check that, in this short planning period case, the optimal plan is exactly in two phases. The firm clears out its inventory right in T. It has not the time to reach the just-in-time regime.

Fig. 4 represents the case of an initial inventory level greater than $\bar{s}_0$. The firm must put down its initial inventory to the level $\bar{s}_0$. It then follows the plan associated to a sales rate equal to w^d. We observe that, in that case, the production phase is much shorter than the selling one.

We have seen from Theorem 9 that, when $s_0 \le \int_0^T \delta(w^a(u))\mathrm{d}u$, the firm can afford to sell out its inventory at some rate which ranges a part of the path of w^a. From Theorem 10 we see that, if the initial inventory is higher than $\int_0^T \delta(w^a(u))\mathrm{d}u$ then the firm must move out its stock faster: the optimal sales rate is given by $x = w^{\hat{r}}$ on $(0, T]$ with $\hat{r} > a$ so that it is greater than w^a on the whole planning period (see Proposition 16 in Section 5 for the increasing feature of w^r with respect to r). In this context, the production may never start. Typically, if $\dot{c}(0) > \dot{\pi}(\infty)$ and $\hat{r} > \dot{\pi}^{-1}(\dot{c}(0))$ then $\dot{\pi}(x) = \dot{\pi}(w^{\hat{r}}) \le \dot{\pi}(\hat{r}) < \dot{c}(0)$ on $(0, T]$ and hence, by definition of g, $y = g(x) = 0$ on $(0, T]$. In any case, the inventory is totally depleted right in T, there is no phase where the firm produces for immediate sales at the constant rate a. This makes qualitatively different the optimal plan obtained on a short planning period and the one obtained on a long period.

From Theorems 9 and 10 we check that there is positive threshold above (below) which the optimal plan is with (without) depletion at time 0. It is given by

$$\bar{s}_0 = \sup_{\{r\in[a,\infty)\mid \tau(r)\le 0,\ \int_0^T \delta(w^r(u))\mathrm{d}u<\infty\}} \int_0^T \delta(w^r(u))\mathrm{d}u.$$

Recall that, by time homogeneity of BW (see Remark 7), the length T_a of the domain of the maximal solution of $BW(a)$ and the threshold $\bar{s}_0$ obtained for long planning periods do not depend on T. To the contrary, when the planning period is short, the threshold may depend on T. Let us denote $(w^r_{T_a}, \tau_{T_a}(r))$ the maximal solution of $BW(r)$ with terminal time T_a, for any $r \in [a, \infty)$. By using Remark 7 we see that, given π, c, s and hence T_a, the dependence of $\bar{s}_0$ with respect to T is given by

$$\bar{s}_0(T) = \sup_{\{r\in[a,\infty)\mid \tau_{T_a}(r)\le 0\,,\ \int_{T_a-T}^{T_a} \delta(w^r_{T_a}(u))\mathrm{d}u<\infty\}} \int_{T_a-T}^{T_a} \delta(w^r_{T_a}(u))\mathrm{d}u$$

It would be of interest to study the monotonicity of $\bar{s}_0$ with respect to T and it would be economically funded to obtain that the shorter the period is, the lower the threshold is. Also, notice that, while we indeed prove the existence of such a threshold, we do not provide some conditions for deciding whether it is finite or infinite. These are directions for future research.

5. Proofs of Theorems 6–10

This section contains the study of the backward integro-differential equation.

5.1. Maximal solutions for BW: Proof of Theorem 6

Let us set $D \triangleq (\dot{\pi}(\infty), \dot{\pi}(a)]$. By Assumption (H) we can define Δ on D by

$$\Delta(v) \triangleq \delta(\dot{\pi}^{-1}(v)) = \dot{\pi}^{-1}(v) - \dot{c}^{-1}(v\mathbf{1}_{v>\dot{c}(0)} + \dot{c}(0)\mathbf{1}_{v\le\dot{c}(0)}), \quad \forall v \in D.$$

For later purpose, we make the following remark.

Remark 11. By Assumptions (H), $(H\pi)$ and (Hc), Δ is nonnegative on D, Lipschitzian on each compact set in D and bounded from above by $\dot{\pi}^{-1}$ on D.

In the sequel, we use the following notation: for all interval I and all set $E \subset \mathbb{R}$, we denote by $C(I; E)$ the set of functions that are continuous on I with values in E. For $E = \mathbb{R}$, we simply write $C(I)$ for $C(I; E)$.

We will obtain Theorem 6 as a corollary of

Theorem 12. *For every $\theta \in D$, there exists a unique couple $(\tau, z) \in [-\infty, T) \times C((\tau, T]; D)$, such that z satisfies the following equation on $(\tau, T]$*

$$\widetilde{BW}(\theta) : z(t) = \theta - \int_t^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right)\right]\mathrm{d}u,$$

and such that,

$$\text{either } \tau = -\infty, \text{ or } \tau > -\infty \text{ and } z(\tau+) = \dot{\pi}(\infty). \tag{11}$$

Moreover, z is increasing.

If Theorem 12 holds then, for every $r \in [a, \infty)$, since $\dot{\pi}(r) \in D$, there exists a unique couple $(\tau, z) \in [-\infty, T) \times C((\tau, T]; D)$ such that

$$z(t) = \dot{\pi}(r) - \int_t^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right)\right]\mathrm{d}u, \quad \forall t \in (\tau, T] \tag{12}$$

and such that the boundary condition (11) holds. Since z is continuous on $(\tau, T]$, by Remark 11 and by Assumption (Hs), we see that the function $u \to \lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right)$ is continuous on $(\tau, T]$. We then deduce from (12) that z is in actual fact continuously differentiable on $(\tau, T]$ with

$$\dot{z}(t) = \lambda z(t) + \dot{s}\left(\int_t^T \Delta(z(v))\mathrm{d}v\right), \quad \forall t \in (\tau, T]. \tag{13}$$

Then, it is easy to check that z satisfies:

$$\mathrm{e}^{-\lambda t} z(t) = \mathrm{e}^{-\lambda T}\dot{\pi}(r) - \int_t^T \mathrm{e}^{-\lambda u}\ \dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right)\mathrm{d}u, \quad \forall t \in (\tau, T]. \tag{14}$$

Under Assumption (H), we can set $w \triangleq \dot{\pi}^{-1} \circ z$ and see that, since z is in $C((\tau, T]; D)$, w is in $C((\tau, T]; [a, \infty))$. From (14) and (11), we deduce that w solves $BW(r)$ on $(\tau, T]$ and satisfies the following boundary condition:

$$\text{either } \ \tau = -\infty, \ \text{ or } \ \tau > -\infty \ \text{ and } \ w(\tau+) = \infty.$$

Besides, since z is increasing, it follows from Assumption (H) again, that w is decreasing. Since $w(T) = r \in [a, \infty)$, it follows that w takes values in $[a, \infty)$ on $(\tau, T]$. Recalling that, by Remark 4, the function δ is continuous, increasing and nonnegative on $[a, \infty)$, we finally obtain that $\delta \circ w$ is continuous, decreasing an nonnegative on $(\tau, T]$. This concludes the proof of Theorem 6.

We now turn to the proof of Theorem 12. It is based on Proposition 14 below. From this proposition, we have the existence of a solution for $\widetilde{BW}(\theta)$ for every terminal condition $\theta \in D$: there exists some continuous function z which satisfies $\widetilde{BW}(\theta)$ on some nonempty interval $(((\gamma, T]]]$. We also see that a solution (γ, z) which does not satisfy the boundary condition (11) can be extended in another solution on some larger interval. Then, the existence of a solution without possible extension, i.e. defined on a maximal interval, will be obtained by an induction argument and Zorn's Lemma. The uniqueness of this maximal solution will follow from the last item of Proposition 14 which states some continuous dependence result of the solution with respect to the terminal condition.

Before providing Proposition 14 let us give a precise definition for the solutions of $\widetilde{BW}$.

Definition 13. *Given $\theta \in D$, we will say that (γ, z) is a solution of $\widetilde{BW}(\theta)$ if: $\gamma \in [-\infty, T)$, $z \in C((\gamma, T]; D)$ and z satisfies the equation $\widetilde{BW}(\theta)$* on $(\gamma, T]$.

In the sequel, for $\phi \in C([a, b])$, we denote $\|\phi\|_{[a,b]} \triangleq \sup_{a \le u \le b} |\phi(u)|$.

Proposition 14.

1. *For every $\theta \in D$, the equation $\widetilde{BW}(\theta)$ has at least one solution. If (γ, z) is such a solution then, z is increasing on $(\gamma, T]$.*
2. *Let (γ, z) be a solution of $\widetilde{BW}(\theta)$ for some $\theta \in D$.* If *$\gamma > -\infty$ and if z satisfies*

   $$z(\gamma+) > \dot{\pi}(\infty)$$

   then $\widetilde{BW}(\theta)$ has a solution $(\hat{\gamma}, \hat{z})$ which satisfies $\hat{\gamma} < \gamma$ and $\hat{z}_{|}(\gamma, T] \equiv z$.
3. *Let $\eta \in (-\infty, T]$ and let L be a compact subset of D. Then, there exists some constant $K_{\eta,L} > 0$ such that if (γ, z) is a solution of $\widetilde{BW}(\theta)$ and (γ', z') is a solution of $\widetilde{BW}(\theta')$ then, for all $t \in [\eta, T]$* with *$t > \gamma, \gamma'$ and such that z and z' both map $[t, T]$ into L we have*:

   $$\|z - z'\|_{[t,T]} \le K_{\eta,L} |\theta - \theta'| \ .$$

Proof. The proof is given in Appendices A and B. It is based on some classical existence result for functional integro-differential equations. See for example Hale and Verduyen Lunel (1993) for a good guide to this literature. □

We are now in position to give the

Proof of Theorem 12. Let us first establish uniqueness. Let (γ, z) and (γ', z') satisfying the requirements of Theorem 12. Fix some arbitrary $t \in (\max(\gamma, \gamma'), T)$. Since z and z' are in $C((\max\{\gamma, \gamma'\}, T]; D)$, there exists some compact $L \subset D$ such that z and z' both map $[t, T]$ into L. It follows from item 3 of Proposition 14 that $z \equiv z'$ on $[t, T]$. By arbitrariness of t in $(\max(\gamma, \gamma'), T]$, we then have $z \equiv z'$ on $(\max(\gamma, \gamma'), T]$, so that $\gamma, \gamma' \geq \max\{\gamma, \gamma'\}$. Therefore $\gamma = \gamma'$ and hence $(\gamma, z) = (\gamma, z')$. This provides uniqueness.

We now prove that existence holds. Define $E \triangleq ((\gamma, z)$ solution of $\widetilde{BW}(\theta))$. By item 1 of Proposition 14, E is not empty. It is easy to check that E is inductive for the order defined by $(\gamma_2, z_2) \succcurlyeq (\gamma_1, z_1) \Leftrightarrow \gamma_2 \leq \gamma_1$ and $z_2 \equiv z_1$ on $(\gamma_1, T]$. Thus, by Zorn's lemma, it admits a maximal element (τ, z). If $\tau = -\infty$, then the proof is concluded. We now assume that $\tau \neq -\infty$. We have to prove that $z(\tau+) = \dot{\pi}(\infty)$. Assume to the contrary that $z(\tau+) > \dot{\pi}(\infty)$. Then, by Proposition 14 there exists some $(\hat{\tau}, \hat{z})$ in E such that $\hat{\tau} < \tau$ and $\hat{z}_{|(\tau,T]} \equiv z$. This contradicts the maximal feature of (τ, z) in E. The proof of Theorem 12 is completed. □

5.2. *Proofs of Theorems 8 and 9*

Proof of Theorem 8. Recall that by Theorem 6 the function $\delta \circ w^a$ is continuous, decreasing and nonnegative on $(\tau(a), T]$. Therefore the function

$$t \in (\tau(a), T] \rightarrow S(t, a) \triangleq \int_t^T \delta(w^a(u))\mathrm{d}u \in \mathbb{R}^+$$

is well-defined, decreasing and continuous on $(\tau(a), T]$. It has a limit as t goes to $\tau(a)$ which is

$$s^a \triangleq \sup_{t \in (\tau(a),T]} \int_t^T \delta(w^a(u))\mathrm{d}u = \int_{\tau(a)}^T \delta(w^a(u))\mathrm{d}u \in \mathbb{R}^+ \cup \{\infty\}$$

Since $\tau(a) < T$, it is positive. Assume that $T \in (T_a, \infty)$, i.e. $\tau(a) > 0$.

1. Recall that $s_0 > 0 = S(T, a)$. So that, if $s_0 \leq s^a = S(\tau(a)+, a)$ the existence of a unique $\tau_0 \in [\tau(a), T)$ such that $s_0 = \int_{\tau_0}^T \delta(w^a(u))\mathrm{d}u$ is insured by the Intermediate-Value Theorem and the above-stated properties of the function $S(\cdot, a)$. In addition, since $\tau(a) > 0$, we have $\tau_0 > 0$. At this point, it is easy to see that, by Theorem 6, w^a satisfies the requirements of Proposition 5 with the couple (a, τ_0). We can therefore conclude that the optimal plan is given by (9). It is regular.
2. Assume that s^a is finite and that $s_0 > s^a$. Since $\tau(a) > 0 > -\infty$, by Theorem 6, we have $w^a(\tau(a)+) = \infty$. Therefore w^a satisfies the requirements of Proposition 5 for the couple $(a, \tau(a))$ and the optimal plan is given by (9). It is not regular. This completes the proof of Theorem 8.

   The proof of Theorem 9 is very similar to item 1 above. It is therefore omitted. □

5.3. *Proof of Theorem 10*

This subsection contains the study of the backward cumulative inventory level at time 0+. As mentioned in Section 4, Theorem 10 relies on the monotonicity, continuity and boundness properties of the map S_0 which associates to each terminal condition r such that $\tau(r) \leq 0$ the backward cumulative inventory level at time 0+ induced by w^r:

$$S_0 : \{r \in [a, \infty) | \tau(r) \leq 0\} \rightarrow \mathbb{R}^+ \cup \{\infty\}, \quad r \mapsto \int_0^T \delta(w^r(u))\,\mathrm{d}u.$$

Our key result for the study of the regularity properties of S_0 is the following corollary of item 3 of Proposition 14, which obviously holds under Assumption $(H\pi)$ and Remark 4.

Corollary 15. *Let $\eta \in (-\infty, T]$ and let L be a compact subset of $[a, \infty)$. Then, there exists some constant $K_{\eta,L} > 0$ such that: if for some $r, r' \in [a, \infty)$ and some $t \in [\eta, T]$ the functions w^r and $w^{r'}$ both map $[t, T]$ into L then, $\|w^r - w^{r'}\|_{[t,T]} + \|\delta(w^r) - \delta(w^{r'})\|_{[t,T]} \le K_{\eta,L}|r - r'|$.*

We will also use the following result on the increasing feature of the maximal solution of $BW(r)$ with respect to r.

Proposition 16. *Fix r and r'* in $[a, \infty)$ *such that $r' < r$. Then, for every $\bar{t} \in [\max\{\tau(r), \tau(r')\}, T)$, we have*: $w^{r'} \le w^r$ *on $(\bar{t}, T]$.*

Proof. For ease of notation, we write w (respectively w') for w^r (respectively $w^{r'}$). Assume to the contrary that the set $\{t \in (\bar{t}, T] | w(t) < w'(t)\}$ is not empty. Then, by continuity of w and w', and since $w(T) = r > r' = w'(T)$, we have $\sigma \triangleq \sup\{t \in (\bar{t}, T] | w(t) < w'(t)\} \in (\bar{t}, T)$ and $w'(\sigma) = w(\sigma)$.

By definition of w and w', we see that, for all $t \in (\bar{t}, \sigma]$,

$$e^{-\lambda t}\dot{\pi}(w(t)) = e^{-\lambda\sigma}\dot{\pi}(w(\sigma)) - \int_t^\sigma e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(v))dv\right)du,$$

$$e^{-\lambda t}\dot{\pi}(w'(t)) = e^{-\lambda\sigma}\dot{\pi}(w'(\sigma)) - \int_t^\sigma e^{-\lambda u}\dot{s}\left(\int_u^T \delta(w'(v))dv\right)du.$$

Since $w'(\sigma) = w(\sigma)$, this implies that, for all $t \in (\bar{t}, \sigma)$,

$$\frac{\dot{\pi}(w(t)) - \dot{\pi}(w'(t))}{\sigma - t} = \frac{e^{\lambda t}}{\sigma - t}\int_t^\sigma e^{-\lambda u}\left[\dot{s}\left(\int_u^T \delta(w'(v))dv\right) - \dot{s}\left(\int_u^T \delta(w(v))dv\right)\right]du.$$

Since the function $u \mapsto e^{-\lambda u}[\dot{s}(\int_u^T \delta(w'(v))dv) - \dot{s}(\int_u^T \delta(w(v))dv)]$ is continuous on $(\bar{t}, T]$, we can let t tend to σ in the above equality to obtain

$$\lim_{t\nearrow\sigma}\left\{\frac{[\dot{\pi}(w(t)) - \dot{\pi}(w'(t))]}{\sigma - t}\right\} = \dot{s}\left(\int_\sigma^T \delta(w'(v))dv\right) - \dot{s}\left(\int_\sigma^T \delta(w(v))dv\right). \tag{15}$$

We claim that the right-hand side is negative, so that, for some $\eta > 0$, $\dot{\pi}(w) - \dot{\pi}(w') < 0$, on $(\sigma - \eta, \sigma)$, and hence, by the increasing feature of $\dot{\pi}$ on $[a, \infty)$, $w > w'$ on $(\sigma - \eta, \sigma)$. This will contradict the definition of σ and finally proves Proposition 16. To see that the above claim holds, recall that δ is increasing on D. Since $\dot{s}$ is increasing, the claim follows from the definition of σ and the fact that the continuous functions w and w' satisfy $w(T) = r > r' = w'(T)$. $\square$

From now on, we assume that $T \in (0, T_a]$ (i.e. $\tau(a) \le 0$) and that $\int_0^T \delta(w^a(u))du < \infty$, and we switch back to the study of S_0 which will provide the proof of Theorem 10. We begin with examining the set $\tau^{-1}(\mathbb{R}^-) = \{r \in [a, \infty) | \tau(r) \le 0\}$ on which S_0 is defined.

Proposition 17. *The function $\tau : r \in [a, \infty) \mapsto \tau(r) \in [-\infty, T)$ is non-decreasing, right-continuous and satisfies $\lim_{r\to\infty} \tau(r) = T$. Therefore, the set $\tau^{-1}(\mathbb{R}^-)$ is a bounded interval with* $\min \tau^{-1}(\mathbb{R}^-) = a$.

Proof. We split it in three steps.

Step 1. We first prove that $\tau : r \in [a, \infty) \mapsto \tau(r) \in [-\infty, T)$ is non-decreasing. Fix r and r' in $[a, \infty)$ such that $r' < r$ and denote w' (respectively w) for $w^{r'}$ (respectively w^r). By Proposition 16, we have

$$w'(t) \le w(t), \quad \forall t \in (\max\{\tau(r'), \tau(r)\}, T]. \tag{16}$$

Assume that $\tau(r') > \tau(r)$. Then $w'(\tau(r')+) = \infty$ and $w(\tau(r')) < \infty$. By letting t tend to $\tau(r')$ in 16 we obtain a contradiction. Therefore $\tau(r') \le \tau(r)$.

Step 2. We now prove that τ is right-continuous as a map from $[a, \infty)$ into $[-\infty, T)$. Since τ is non-decreasing it suffices to prove that $\inf_{r>r'} \tau(r) = \tau(r')$ for all$r' \in [a, \infty)$. Assume to the contrary that there exists some $r' \in [a, \infty)$ such that

$$\ell \triangleq \inf_{r>r'} \tau(r) > \tau(r') \in [-\infty, T). \tag{17}$$

For ease of notation, we write w' for $w^{r'}$. Since $\ell > \tau(r')$, $w'(\ell)$ is finite and we can fix some real number M such that

$$M > w'(\ell). \tag{18}$$

Since, by Theorem 6, w' is decreasing, we have $M > w'(\ell) > r'$. Therefore, there exists some $r \in [a, M)$ such that

$$0 < r - r' < \frac{M - w'(\ell)}{K_{\ell,[a,M]}}, \tag{19}$$

where $K_{\ell,[a,M]}$ is given by Corollary 15. Moreover, since τ is non-decreasing and by 17, we have: $\tau(r) \geq \inf_{z>r'} \tau(z) = \ell > -\infty$. It follows from Theorem 6 that $w^r(\tau(r)+) = \infty$. Now, since w^r is continuous, decreasing and satisfies $w^r(T) = r < M < \infty$, by the Intermediate-Value Theorem there exists some $t_M \in (\tau(r), T)$ such that $w^r(t_M) = M$. Observing that

$$\ell \leq \tau(r) < t_M, \tag{20}$$

we deduce from the decreasing feature of w' together with 19 that

$$w^r(t_M) - w'(t_M) > M - w'(\ell) > K_{\ell,[a,M]}(r - r'). \tag{21}$$

We now put in light the required contradiction. Since w^r and w' are decreasing, it follows from the equality $w^r(t_M) = M$, (20) and (18), that they both map the interval $[t_M, T]$ into $[a, M]$. Therefore, applying Corollary 15 with $\eta = \ell$ and $L = [a, M]$ we have

$$0 \leq w^r(t_M) - w'(t_M) \leq K_{\ell,[a,M]}(r - r'),$$

which combined with (21) yields the required contradiction.

Step 3. We finally prove that $\lim_{r\to\infty} \tau(r) = T$. Assume to the contrary that

$$\bar{T} \triangleq \lim_{r\to\infty} \tau(r) < T. \tag{22}$$

Considering the sequence $(w^n)_{n\geq a}$, we see that

$$\mathrm{e}^{-\lambda \bar{T}} \dot{\pi}(w^n(\bar{T})) - \mathrm{e}^{-\lambda T} \dot{\pi}(n) = -\int_{\bar{T}}^{T} \mathrm{e}^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(u))\mathrm{d}u \right) \mathrm{d}t.$$

Since, by Theorem 6, w^n is decreasing, we have $w^n(\bar{T}) \geq n$, $\forall n \geq a$. Then letting n tend to ∞ in the previous inequality shows that

$$0 \leq (\mathrm{e}^{-\lambda \bar{T}} - \mathrm{e}^{-\lambda T})\dot{\pi}(\infty) = \lim_{n\to\infty} \left\{ -\int_{\bar{T}}^{T} \mathrm{e}^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(u))\mathrm{d}u \right) \mathrm{d}t \right\} \tag{23}$$

where the inequality holds because $\bar{T} < T$ and $\dot{\pi}(\infty) \geq 0$. In order to obtain a contradiction, we will prove that the right-hand side is negative. Observe that, since $\dot{s}$ is nonnegative,

$$-\int_{\bar{T}}^{T} \mathrm{e}^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(u))\mathrm{d}u \right) \mathrm{d}t \leq -\int_{\bar{T}}^{(\bar{T}+T)/2} \mathrm{e}^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(u))\mathrm{d}u \right) \mathrm{d}t. \tag{24}$$

Recall that w^n is decreasing and satisfies $w^n(T) = n$. By Remark 4, we deduce that there exists some $n_0 \geq a$ such that for all $n \geq n_0$, $\delta(w^n(u)) \geq \delta(w^n(T)) \geq 2$ for all $u \in [(\bar{T} + T)/2, T]$. Therefore, for every $n \geq n_0$ and $t \in [\bar{T}, (\bar{T} + T)/2]$

$$\int_t^T \delta(w^n(u))\mathrm{d}u \geq \int_{(\bar{T}+T)/2}^T \delta(w^n(u))\mathrm{d}u \geq 2 \left(T - \frac{\bar{T} + T}{2} \right) \geq T - \bar{T},$$

where we used the nonnegativity of δ. By the increasing feature and nonnegativity of $\dot{s}$, this implies that for every $n \geq n_0$,

$$-\int_{\bar{T}}^{(\bar{T}+T)/2} \mathrm{e}^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(u)) \mathrm{d}u \right) \mathrm{d}t \leq -\dot{s}(T-\bar{T}) \int_{\bar{T}}^{(\bar{T}+T)/2} \mathrm{e}^{-\lambda t} \mathrm{d}t. \tag{25}$$

From (24) and (25) we deduce that

$$\lim_{n\to\infty} \left\{ -\int_{\bar{T}}^{T} \mathrm{e}^{-\lambda t} \dot{s} \left(\int_t^T \delta(w^n(u)) \mathrm{d}u \right) \mathrm{d}t \right\} \leq -\dot{s}(T-\bar{T}) \int_{\bar{T}}^{(\bar{T}+T)/2} \mathrm{e}^{-\lambda t} \mathrm{d}t < 0, \tag{26}$$

where the last inequality holds because $\bar{T} < T$ and $\dot{s}$ is increasing, nonnegative and therefore positive on $(0, \infty)$. The required contradiction follows from (23) and (26).

The second assertion Proposition 17 is a direct consequence of the above steps. The proof is completed. □

Let us now turn to the monotonicity, continuity and boundness properties of S_0.

Proposition 18.

(1) *The function S_0 is well-defined, non-decreasing and left-continuous as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$.*
(2) *The set $D_0 = \{r \in \tau^{-1}(\mathbb{R}^-) | S_0(r) \in \mathbb{R}^+\}$ is a nonempty bounded interval with minimum equal to a. Let $d \triangleq \sup D_0$. We have $\tau(d) \geq 0$.*
(3) *The function S_0 is continuous and increasing on its domain D_0.*
(4) *If $\tau(d) > 0$ then $\sup_{D_0} S_0 = \infty$.*

Proof. We will make use of the following remark.

Remark 19. By Theorem 6, for any $r \in [a, \infty)$, the function $\delta \circ w^r$ is continuous, decreasing and nonnegative on $(\tau(r), T]$. Therefore, for any $r \in [a, \infty)$, the function:

$$t \in (\tau(r), T] \to S(t, r) \triangleq \int_t^T \delta(w^r(u)) \mathrm{d}u \in \mathbb{R}^+$$

is well-defined, decreasing and continuous on $(\tau(r), T]$. In addition, it has a limit as t goes to $\tau(r)$ which is

$$\sup_{t \in (\tau(r), T]} \int_t^T \delta(w^r(u)) \mathrm{d}u = \int_{\tau(r)}^T \delta(w^r(u)) \mathrm{d}u \in \mathbb{R}^+ \cup \{\infty\}.$$

1. Thus, for any $r \in [a, \infty)$ such that $\tau(r) \leq 0$ we have $S_0(r) = \int_0^T \delta(w^r(u)) \mathrm{d}u \in \mathbb{R}^+ \cup \{\infty\}$. The function S_0 is thus well-defined as a map from $\tau^{-1}(\mathbb{R}^-)$ into $\mathbb{R}^+ \cup \{\infty\}$. Besides, from Proposition 16 it follows that it is non-decreasing.

   Let us prove that it is left-continuous. Since S_0 is non-decreasing, the limit $\lim_{r \nearrow r'} S_0(r)$ exists in $\mathbb{R}^+ \cup \{\infty\}$ for all $r' \in \tau^{-1}(\mathbb{R}^-)$. We will prove that

$$\lim_{r \nearrow r'} S_0(r) = S_0(r') \ \text{in} \mathbb{R}^+ \cup \{\infty\}, \quad \forall r' \in \tau^{-1}(\mathbb{R}^-) \cap (a, \infty). \tag{27}$$

   Fix $r' \in \tau^{-1}(\mathbb{R}^-) \cap (a, \infty)$. We first consider the case $S_0(r' \in \mathbb{R}^+)$. Fix $\varepsilon > 0$. By Remark 19, there exists some $t_\varepsilon \in (0, T]$ such that

$$0 \leq S_0(r') - S(t_\varepsilon, r') < \frac{\varepsilon}{3}. \tag{28}$$

   Let $r \in \tau^{-1}(\mathbb{R}^-)$ such that $r < r'$. Then, by Proposition 16 we have: $w^r \leq w^{r'}$ on $[t_\varepsilon, T]$. Therefore, w^r and $w^{r'}$ both map $[t_\varepsilon, T]$ into $[a, w^{r'}(t_\varepsilon)]$. Then, by Corollary 15, for $\eta = 0$ and $L = [a, w^{r'}(t_\varepsilon)]$, there exists some $K > 0$ such that

$$0 \leq \delta(w^{r'}) - \delta(w^r) \leq K(r' - r) \ \text{on} \ [t_\varepsilon, T], \tag{29}$$

where the first inequality follows from the increasing feature of δ. Since S_0 is non-decreasing, we also have

$$0 \le S_0(r) - S(t_\varepsilon, r) \le S_0(r') - S(t_\varepsilon, r) < \frac{\varepsilon}{3}. \tag{30}$$

Now, defining $\zeta \triangleq \varepsilon/3(T - t_\varepsilon)K$ and using (28)–(30), we see that for all $r \in [r' - \zeta, r')$,

$$\begin{aligned}
&|S_0(r') - S_0(r)| \le |S_0(r') - S(t_\varepsilon, r')| + |S(t_\varepsilon, r') - S(t_\varepsilon, r)| + |S(t_\varepsilon, r) - S_0(r)| \\
&\le \frac{2\varepsilon}{3} + \int_{t_\varepsilon}^{T} [\delta(w^{r'}(u)) - \delta(w^r(u))]\mathrm{d}u \\
&\le \frac{2\varepsilon}{3} + (T - t_\varepsilon)K(r' - r) \\
&\le \varepsilon.
\end{aligned}$$

This proves (27) when $S_0(r') < \infty$.

We now consider the case where $S_0(r') = \infty$. Assume to the contrary that $M \triangleq \lim_{r \nearrow r'} S_0(r) < \infty$. Since S_0 is non-decreasing, we have $M = \sup_{r<r'} S_0(r)$. In order to obtain a contradiction, we will find some $r \in \tau^{-1}(\mathbb{R}^-)$ such that

$$r < r' \quad \text{and} \quad S_0(r) > M. \tag{31}$$

First notice that since $S_0(r') = \infty$ we necessarily have $\tau(r') = 0$. Then, by Remark 19, there exists some $t_M \in (0, T]$ such that $S(t_M, r') > 2M$. Let $K \triangleq K_{0,[a,w(t_M)]}$ be given by Corollary 15. We set: $r \triangleq \max\{r' - M/K(T - t_M), a\}$. By construction, $r < r'$ because by assumption $r' > a$. Then, by the same arguments as in the previous case, we can apply Corollary 15 for $\eta = 0$ and $L = [a, w^{r'}(t_M)]$ on $[t_M, T]$ to obtain

$$\begin{aligned}
S(t_M, r) &= S(t_M, r) - S(t_M, r') + S(t_M, r') \\
&\ge \int_{t_M}^{T} [\delta(w^r(u)) - \delta(w^{r'}(u))]\mathrm{d}u + S(t_M, r') \\
&\ge (T - t_M)K(r - r') + S(t_M, r') \\
&> -M + 2M = M.
\end{aligned}$$

This proves (31) and therefore (27) when $S_0(r') = \infty$. The proof of item 1 is completed.

2. Since S_0 is non-decreasing, its domain D_0 is an interval. Recall that we have assumed that $\tau(a) \le 0$ and $S_0(a) = \int_0^T \delta(w^a(u))\mathrm{d}u < \infty$. Therefore D_0 is nonempty and $\min D_0 = a$. Since D_0 is a subset of $\tau^{-1}(\mathbb{R}^-)$ which is bounded from above, we can define $d \triangleq \sup D_0 \in \mathbb{R}^+$. Assume that $\tau(d) < 0$ then, since by Proposition 17, the function τ is right-continuous, there exists some $d' > d$ such that $\tau(d') < 0$ and hence such that $\int_0^T \delta(w^{d'}(u))\mathrm{d}u < \infty$, so that d' is in D_0 and $d' > \sup D_0$. This is a contradiction. Therefore $\tau(d) \ge 0$.
3. We now prove that S_0 is continuous on its domain D_0. Recall that D_0 is an interval. Since S_0 is left-continuous, we only have to prove that

$$\text{if } r' \in D_0 \text{ and } r' < \sup D_0 \text{ then } \lim_{r \searrow r'} S_0(r) = S_0(r').$$

Let $r' \in D_0$, $r' < \sup D_0$. Then there exists $\bar{r} > r'$ such that $\tau(\bar{r}) \le 0$ and $S_0(\bar{r}) < \infty$. Fix $\varepsilon > 0$. By Remark 19, there exists some $t_\varepsilon \in (0, T]$ such that $0 \le S_0(\bar{r}) - S(t_\varepsilon, \bar{r}) < \varepsilon/3$. Now, for all $r \in (r', \bar{r}]$, by Proposition 16 and by the increasing feature of δ we have

$$0 \le S_0(r') - S(t_\varepsilon, r') \le S_0(r) - S(t_\varepsilon, r) \le S_0(\bar{r}) - S(t_\varepsilon, \bar{r}) < \frac{\varepsilon}{3}.$$

Notice that by Proposition 16 again, for every $r \in [r', \bar{r}]$, w^r maps $[t_\varepsilon, T]$ into $[a, w^{\bar{r}}(t_\varepsilon)]$. The proof can therefore be completed by applying Corollary 15, with $\eta = 0$ and $L = [a, w^{\bar{r}}(t_\varepsilon)]$, to $w^{r'}$ and w^r on $[t_\varepsilon, T]$. We have proved that S_0 is continuous on D_0.

Let us now prove that S_0 is increasing on D_0. Fix some $r, r' \in D_0$ such that $r < r'$. Since $w^r(T) = r < r' = w^{r'}(T)$, by continuity we have $w^r < w^{r'}$ on a subset of $(0, T]$ which has a positive Lebesgue measure. Besides, by Proposition 16, $w^r \leq w^{r'}$ on $(0, T]$. Therefore, since the function δ is increasing, we have

$$S_0(r) - S_0(r') = \int_0^T [\delta(w^r(u)) - \delta(w^{r'}(u))]\mathrm{d}u > 0.$$

This ends the proof of item 3.

4. We now prove that if $\tau(d) > 0$ then $\sup_{D_0} S_0 = \infty$. We claim that

$$\lim_{r \nearrow d} w^r(\tau(d)) = \infty. \tag{32}$$

Therefore, there exists some sequence (r_n) in D_0 increasing to d such that

$$\lim_{n \to \infty} w^{r_n}(\tau(d)) = \infty. \tag{33}$$

Using the fact that $\delta \circ w^{r_n}$ is nonnegative and decreasing, we see that

$$S_0(r_n) = \int_0^T \delta(w^{r_n}(u))\mathrm{d}u \geq \int_0^{\tau(d)} \delta(w^{r_n}(u))\mathrm{d}u \geq \tau(d)\delta(w^{r_n}(\tau(d))).$$

The proof is completed by using (33) and recalling that $\tau(d) > 0$ and that $\lim_{r \to \infty} \delta(r) = \infty$ (see Remark 4).

We now prove (32). Observe that, by Proposition 16, the function $r \mapsto w^r(\tau(d))$ is non-decreasing on $[a, d)$. We therefore have $M \triangleq \lim_{r \nearrow d} w^r(\tau(d)) = \sup_{r<d} w^r(\tau(d))$ and $M \in [a, \infty]$. We will prove that $M = \infty$. Assume to the contrary that $M = \sup_{r<d} w^r(\tau(d)) < \infty$. In order to have a contradiction, we will find some $r' \in [a, d)$ such that

$$w^{r'}(\tau(d)) > M. \tag{34}$$

First notice that, since $\tau(d) > 0 > -\infty$, we have by Theorem 6, $w^d(\tau(d)+) = \infty$. Therefore, since $w^d(T) = d$, by continuity, there exists some $t_M \in (\tau(d), T]$ such that

$$w^d(t_M) = 2\max\{M, d\}. \tag{35}$$

Let $K \triangleq K_{\tau(d),[a,2\max\{M,d\}]}$ be given by Corollary 15 and define

$$r' \triangleq \max\left\{\left(d - \frac{M}{2K}\right), a\right\}. \tag{36}$$

By definition, $r' \in [a, \infty)$. Moreover, since by assumption $\tau(d) > 0 \geq \tau(a)$ and since τ is non-decreasing, we have $d > a$. Therefore, $r' \in [a, d)$.

We now prove (34). Since w^d is decreasing and $w^d(t_M) = 2\max\{M, d\}$, w^d maps $[t_M, T]$ into $[a, 2\max\{M, d\}]$. Besides, since $r' < d$, it follows from Proposition 16 that $w^{r'} \leq w^d$ on $[t_M, T]$. Hence, $w^{r'}$ also maps $[t_M, T]$ into $[a, 2\max\{M, d\}]$. Therefore, with $\eta = \tau(d)$ and $L = [a, 2\max\{M, d\}]$ in Corollary 15, applied to w^d and $w^{r'}$, we have

$$|w^{r'}(t_M) - w^d(t_M)| \leq K|r' - d|, \quad \text{which reads, } w^{r'}(t_M) - w^d(t_M) \geq K(r' - d).$$

By (35), we therefore have $w^{r'}(t_M) \geq 2\max\{M, d\} + K(r' - d)$. Noticing that, by (36), $K(r' - d) \geq -M/2$, we obtain $w^{r'}(t_M) \geq 2M - M/2 > M$. Since $t_M > \tau(d)$ and $w^{r'}$ is decreasing, this implies that $w^{r'}(\tau(d)) > M$, which ends the proof of (34) and therefore proves (32). The proof of Proposition 18 is completed.

We are now in position to complete the

Proof of Theorem 10. Recall that $\tau(a) \le 0$ and $\int_0^T \delta(w^a(u))\mathrm{d}u < \infty$.

1. By Proposition 18 the function S_0 is continuous and increasing on its domain D_0. Assume that $\sup_{D_0} S_0 = \infty$. Recall that $\min D_0 = a$. Then for $s_0 > \int_0^T \delta(w^a(u))\mathrm{d}u$, by the Intermediate-Value Theorem and since S_0 is increasing on D_0, there exists a unique $\hat{r} \in D_0$, $\hat{r} < \sup D_0 = d$ such that $s_0 = \int_0^T \delta(w^{\hat{r}}(u))\mathrm{d}u$. It is then easy to see that $w^{\hat{r}}$ satisfies the requirements of Proposition 5 for the couple $(r, \tau_0) = (\hat{r}, 0)$ and hence that the optimal plan is regular given by: $(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}}))$.
2. Assume that $\sup_{D_0} S_0 < \infty$. From item 4 of Proposition 18 it follows that $\tau(d) = 0$. Besides, since S_0 is non-decreasing and left-continuous on $\tau^{-1}(\mathbb{R}^-)$ we have

$$\sup_{D_0} S_0 = \lim_{\substack{r \to d \\ r < d}} S_0(r) = S_0(d) = \int_0^T \delta(w^d(u))\mathrm{d}u$$

and hence, $\int_0^T \delta(w^d(u))\mathrm{d}u < \infty$ and $D_0 = [a, d]$. Recalling that S_0 is continuous and increasing on its D_0 we see that, if $s_0 \in [S_0(a), S_0(d)]$ then there exists a unique $\hat{r} \in (a, d]$ such that $s_0 = \int_0^T \delta(w^{\hat{r}}(u))\mathrm{d}u$. We can then conclude as in the previous case that the optimal plan is regular given by $(\alpha, x, y) = (0, w^{\hat{r}}, g(w^{\hat{r}}))$.

Consider now $s_0 > \int_0^T \delta(w^d(u))\mathrm{d}u$. Since $\tau(d) = 0 > -\infty$, we have by Theorem 6, $w^d(0+) = \infty$. The function w^d thus satisfies the requirements of Proposition 5 for the couple $(d, 0)$. The optimal plan is then given by $(\alpha, x, y) = \left(s_0 - \int_0^T \delta(w^d(u))\mathrm{d}u, w^d, g(w^d)\right)$. It is not regular. The proof of Theorem 10 is completed. $\square$

Acknowledgement

We thank an anonymous referee for providing comments that improved the exposition of this paper.

Appendix A. Proof of Proposition 5

Let w and (r, τ_0) be as in the statement. We will prove that the plan $(\tilde{\alpha}, \tilde{x}, \tilde{y})$ defined by

$$\tilde{\alpha} \triangleq s_0 - \int_{\tau_0}^T \delta(w(u))\mathrm{d}u' \tag{A.1}$$

$$\tilde{x} \triangleq w(\cdot + \tau_0)\mathbf{1}_{(0,T-\tau_0]} + a\mathbf{1}_{(T-\tau_0,T]} \tag{A.2}$$

$$\tilde{y} \triangleq (\tilde{x}) \tag{A.3}$$

is in $\mathcal{B}$ and satisfies conditions (I)–(VI) of Proposition 3 which characterize (α, x, y).

Since w is decreasing on $(\tau_0, T]$ and $w(T) = r \in [a, \infty)$, by definition (see (A.2)) the function $\tilde{x}$ is non-increasing and takes values in $[a, \infty)$ on $(0, T]$. Recall that by Remark 4, the function g is non-increasing, taking values in $[0, a]$ on $[a, \infty)$. Therefore, by (A.3) the function $\tilde{y}$ is non-decreasing and takes values in $[0, a]$ on $(0, T]$. Besides, by (8), we have $\int_{\tau_0}^T \delta(w(u))\mathrm{d}u \le s_0$ i.e. $\int_{\tau_0}^T [w(u) - g(w(u))]\mathrm{d}u \le s_0$ and hence, since g takes values in $[0, a]$ and $\tau_0 \ge 0$ we have $\int_{\tau_0}^T w(u)\mathrm{d}u \le s_0 + Ta$. Since $w \ge 0$, this implies that $w \in L^1_+[\tau_0, T]$ and by construction that $\tilde{x} in L^1_+[0, T]$. By (8) again, we have $\tilde{\alpha} \ge 0$. Thus $(\tilde{\alpha}, \tilde{x}, \tilde{y}) \in \mathbb{R}^+ \times L^1_+[0, T] \times L^1_+[0, T]$ and $\tilde{x}$ and $\tilde{y}$ satisfy (I).

Let us now prove that

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})} \ge 0 \text{ on } (0, T] \text{ and that (II), (IIIa), (IIIb), and (IV) hold.} \tag{A.4}$$

We will obtain that $\tilde{T}_0 \triangleq \inf\{t \in (0, T] | S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = 0\} = T - \tau_0$. We begin with proving that $\tilde{x} = \tilde{y} = a$ and $S^{(\tilde{\alpha},\tilde{x},\tilde{y})} = 0$ on $(T - \tau_0, T]$. By definition (see (A.2) and (A.3)) and since $g(a) = a$ we have

$$\tilde{x} = a = g(a) = g(\tilde{x}) = \tilde{y} \text{ on } (T - \tau_0, T], \tag{A.5}$$

so that

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(T - \tau_0), \quad \text{for all } t \in (T - \tau_0, T]. \tag{A.6}$$

By (A.1)–(A.3) and after a change of variable, for any $t \in (0, T - \tau_0]$, we have

$$\begin{aligned} S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) &= s_0 - \tilde{\alpha} + \int_0^t [\tilde{y}(u) - \tilde{x}(u)]\mathrm{d}u \\ &= s_0 - \tilde{\alpha} + \int_0^t [g(\tilde{x}(u)) - \tilde{x}(u)]\mathrm{d}u \\ &= \int_{\tau_0}^T \delta(w(u))\mathrm{d}u + \int_0^t [g(w(u + \tau_0)) - w(u + \tau_0)]\mathrm{d}u \\ &= \int_{\tau_0}^T \delta(w(u))\mathrm{d}u + \int_{\tau_0}^{t+\tau_0} [g(w(u)) - w(u)]\mathrm{d}u, \end{aligned}$$

that is, by definition of δ,

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = \int_{t+\tau_0}^T \delta(w(u))\mathrm{d}u \ . \tag{A.7}$$

Therefore $S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(T - \tau_0) = 0$ and by (A.6)

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = 0, \quad \text{for all } t \in [T - \tau_0, T]. \tag{A.8}$$

In order to complete the proof of (A.4) it suffices to check that

$$S^{(\tilde{\alpha},\tilde{x},\tilde{y})} \text{ is decreasing on } (0, T - \tau_0]. \tag{A.9}$$

Indeed from (A.8) and since by assumption $\tau_0 < T$ we will therefore deduce that $S^{(\tilde{\alpha},\tilde{x}\tilde{y})} \geq 0$ on $(0, T]$, $\tilde{T}_0 \triangleq \inf\{t \in (0, T] | S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(t) = 0\} = T - \tau_0$ and $\tilde{T}_0 > 0$, so that (II) and (IIIa) hold by (A.8) again, (IIIb) holds by (A.5) and (IV) holds by (A.9).

In order to prove that $S^{(\tilde{\alpha},\tilde{x},\tilde{y})}$ is decreasing on $(0, T - \tau_0]$, recall that by Remark 4, δ takes positive values on (a, ∞). Moreover, since by (i) the function w is decreasing on $(\tau_0, T]$ and satisfies $w(T) = r \geq a$, it takes values in (a, ∞) on (τ_0, T). Consequently, $\delta \circ w$ is positive on (τ_0, T). It then follows from (A.7) that $S^{(\tilde{\alpha},\tilde{x},\tilde{y})}$ is decreasing on $(0, T - \tau_0]$. This concludes the proof of (A.4).

Let us now check that (IIIc) is satisfied, i.e. that if $\tilde{T}_0 < T$ then $\tilde{x}(\tilde{T}_0-) = a$. Observe that, by (i) and (A.2) we have $\tilde{x}(\tilde{T}_0-) = w(T-) = w(T) = r \in [a, \infty)$. Hence, since $[\tilde{T}_0 < T \Leftrightarrow \tau_0 > 0]$ and since $(r, \tau_0) \in \{(a, \infty) \times \{0\}\} \cup \{\{a\} \times [0, T)\}$ (see (6)) condition (IIIc) holds.

Let us now turn to the proof of (V). By construction (see (A.3)) $\tilde{y} = g(\tilde{x})$ on $[0, T]$ so that (4) holds. Since w is continuous on $(\tau_0, T]$, by construction $\tilde{x}$ is continuous on $(0, T - \tau_0] = (0, \tilde{T}_0]$ and so is $\tilde{y} = g(\tilde{x})$ (recall that g is continuous on $[a, \infty)$).

Let us establish Eq. (3). Since by (A.2) $w = x(\cdot - \tau_0)$ on $(\tau_0, T]$, Eq. (7) reads:

$$\mathrm{e}^{-\lambda t}\dot{\pi}(\tilde{x}(t - \tau_0)) = \mathrm{e}^{-\lambda T}\dot{\pi}(r) - \int_t^T \mathrm{e}^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(v))\mathrm{d}v\right)\mathrm{d}u, \quad \forall t \in (\tau_0, T],$$

which, after a change of variable from $(\tau_0, T]$ onto $(0, T - \tau_0] = (0, \tilde{T}_0]$ yields:

$$\mathrm{e}^{-\lambda(t+\tau_0)}\dot{\pi}(\tilde{x}(t)) = \mathrm{e}^{-\lambda T}\dot{\pi}(r) - \int_{t+\tau_0}^T \mathrm{e}^{-\lambda u}\dot{s}\left(\int_u^T \delta(w(v))\mathrm{d}v\right)\mathrm{d}u, \quad \forall t \in (0, \tilde{T}_0].$$

Equivalently,

$$\mathrm{e}^{-\lambda t}\dot{\pi}(\tilde{x}(t)) = \mathrm{e}^{-\lambda(T-\tau_0)}\dot{\pi}(r) - \int_{t+\tau_0}^{T} \mathrm{e}^{-\lambda(u-\tau_0)}\dot{s}\left(\int_u^T \delta(w(v))\mathrm{d}v\right)\mathrm{d}u, \quad \forall t \in (0, \tilde{T}_0].$$

Using a change of variable in the integral and since $T - \tau_0 = \tilde{T}_0$ we obtain

$$\mathrm{e}^{-\lambda t}\dot{\pi}(\tilde{x}(t)) = \mathrm{e}^{-\lambda\tilde{T}_0}\dot{\pi}(r) - \int_t^{\tilde{T}_0} \mathrm{e}^{-\lambda u}\dot{s}\left(\int_{u+\tau_0}^T \delta(w(v))\mathrm{d}v\right)\mathrm{d}u, \quad \forall t \in (0, \tilde{T}_0].$$

Plugging (A.7) in this last equation we get

$$\mathrm{e}^{-\lambda t}\dot{\pi}(\tilde{x}(t)) = \mathrm{e}^{-\lambda\tilde{T}_0}\dot{\pi}(r) - \int_t^{\tilde{T}_0} \mathrm{e}^{-\lambda u}\dot{s}\left(S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(u)\right)\mathrm{d}u, \quad \forall t \in (0, \tilde{T}_0]. \tag{A.10}$$

Observe that since $\tilde{x}$ is non-increasing and takes values in $[a, \infty)$ and since $\dot{\pi}$ is continuous and decreasing on $[a, \infty)$ and satisfies $\lim_{x\to\infty}\dot{\pi}(x) = \dot{\pi}(\infty)$ we have $\lim_{t\searrow 0}\dot{\pi}(\tilde{x}(t)) = \dot{\pi}(\tilde{x}(0+)) \in [\dot{\pi}(\infty), \dot{\pi}(a)]$. Therefore, sending t to 0 in (A.10) we obtain

$$\dot{\pi}(\tilde{x}(0+)) = \mathrm{e}^{-\lambda\tilde{T}_0}\dot{\pi}(r) - \int_0^{\tilde{T}_0} \mathrm{e}^{-\lambda u}\dot{s}\left(S^{(\tilde{\alpha},\tilde{x},\tilde{y})}(u)\right)\mathrm{d}u \quad . \tag{A.11}$$

By computing the difference between (A.10) and (A.11), we obtain (3).

To complete the proof of Proposition 5, it remains to show that (VI) is satisfied, i.e. to check that if $\tilde{\alpha} > 0$ then $\tilde{x}(0+) = \infty$.

By (A.2) we have $\tilde{x}(0+) = w(\tau_0+)$ and by (A.1) $\tilde{\alpha} = s_0 - \int_{\tau_0}^T \delta(w(u))\mathrm{d}u$ so that the above assertion holds since it is equivalent to condition (8). This ends the proof of Proposition 5.

Appendix B. Proof of Proposition 14

B.1. Proof of Proposition 14 item 1

Since Δ and $\dot{s}$ are continuous on respectively D and $\mathbb{R}^+$, for $\theta \in D$, finding a solution for $\widetilde{BW}(\theta)$ amounts to find some (γ, z) such that $\gamma \in [-\infty, T)$, $z \in C((\gamma, T]; D)$ and z satisfies:

$$\dot{z}(t) = \lambda z(t) + \dot{s}\left(\int_t^T \Delta(z(u))\mathrm{d}u\right) \quad \text{for } t \in (\gamma, T] \text{ and } z(T) = \theta. \tag{B.1}$$

In order to use a standard existence result in Hale and Verduyen Lunel (1993), we rewrite (B.1) using the terminology of retarded equations. For this purpose, let us first introduce some notation.

Notation Let $\rho > 0$ and $A \geq 0$. We set $C_\rho \triangleq C([0, \rho]\ \mathbb{R})$. For every $z \in C([T - A, T + \rho]\ \mathbb{R})$ and for any $t \in [T - A, T]$ we let z_t be defined by $z_t(u) = z(t + u)$ for $u \in [0, \rho]$.

With this notation we see that, if one find some (γ, z) such that $\gamma \in [T - \rho, T)$, $z \in C((\gamma, T]; D)$ and z satisfies:

$$\dot{z}(t) = \lambda z_t(0) + \dot{s}\left(\int_0^{T-t} \Delta(z_t(u))\mathrm{d}u\right) \quad \text{for } t \in (\gamma, T] \text{ and } z_T(0) = \theta. \tag{B.2}$$

then one finds some solution for $\widetilde{BW}(\theta)$.

Let us now state a backward version of Theorem 2.1, p. 44 in Hale and Verduyen Lunel (1993) from which we will obtained the desired existence result.

Theorem 20. *Fix $\rho \geq 0$. Let C_ρ be endowed with the uniform topology and let Ω be an open subset in $\mathbb{R} \times C_\rho$. If F is defined, continuous on Ω, valued in $\mathbb{R}$, then for every $(\eta, \theta) \in \Omega$ there exist $A > 0$ and $z \in C((\eta - A, \eta + \rho], \mathbb{R})$ such that $(t, z_t) \in \Omega$* and *$\dot{z}(t) = F(t, z_t)$ for every $t \in (\eta - A, \eta]$ and $z_\eta \equiv \theta$ on* $[0, \rho]$.

Let us consider the following extension of the function Δ on $(\dot{\pi}(\infty), \infty)$:

$$\Delta(v) = \begin{cases} \dot{\pi}^{-1}(v) - \dot{c}^{-1}(v\mathbf{1}_{v>\dot{c}(0)} + \dot{c}(0)\mathbf{1}_{v\leq\dot{c}(0)}) & \text{if } v \in (\dot{\pi}(\infty), \dot{\pi}(a)] \\ 0 & \text{if } v \in (\dot{\pi}(a), \infty) \end{cases}$$

Let us set $\Omega = (T-\rho, T+\rho) \times C([0,\rho]; (\dot{\pi}(\infty), \infty))$, for some $\rho > 0$. Let F be defined on Ω by

$$F(t,\phi) = \lambda\phi(0) + \dot{s}\left(\int_0^{T-t} \Delta(\phi(u))\mathrm{d}u\mathbf{1}_{t\le T}\right).$$

By assumptions (H), $(H\pi)$ and (Hc) and by construction, one can see that Δ is nonnegative on $(\dot{\pi}(\infty), \infty)$, Lipschitzian on each compact set in $(\dot{\pi}(\infty), \infty)$ and bounded from above by $\dot{\pi}^{-1}$ on $(\dot{\pi}(\infty), \dot{\pi}(a)]$. Using this properties and the fact that $\dot{s}$ is Lipschitzian on every compact in $\mathbb{R}^+$, we prove that F is continuous on Ω. Let $(t_n, \phi_n)_n$ be a sequence in Ω converging to $(t,\phi) \in \Omega$. We have

$$|F(t_n,\phi_n) - F(t,\phi)| \le \lambda|\phi_n(0) - \phi(0)| + \left|\dot{s}\left(\int_0^{T-t_n} \Delta(\phi_n(u))\mathrm{d}u\mathbf{1}_{t_n\le T}\right) - \dot{s}\left(\int_0^{T-t} \Delta(\phi(u))\mathrm{d}u\mathbf{1}_{t\le T}\right)\right| \quad \text{(B.3)}$$

Since ϕ is continuous on $[T-\rho, T+\rho]$, it is bounded. Moreover, $(\phi_n)_n$ converges uniformly to ϕ in $C([0,\rho]; (\dot{\pi}(\infty), \infty))$. Therefore there exist some compact subset K in $(\dot{\pi}(\infty), \infty)$ and some $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$, ϕ and ϕ_n are valued in K. From the fact that Δ is nonnegative and continuous on $(\dot{\pi}(\infty), \infty)$ and since the sequence $(t_n)_n$ is bounded, we then deduce that, for every $n \ge n_0$, the terms $\int_0^{T-t_n} \Delta(\phi_n(u))\mathrm{d}u\mathbf{1}_{t_n\le T}$ and $\int_0^{T-t} \Delta(\phi(u))\mathrm{d}u\mathbf{1}_{t\le T}$ are in the same compact subset of $\mathbb{R}^+$. Thus, since, by assumption (Hs), $\dot{s}$ is Lipschitzian on every compact subset in $\mathbb{R}^+$, there exists some constant $C \ge 0$ such that we have

$$\begin{aligned}\forall n \ge n_0,\ &\left|\dot{s}\left(\int_0^{T-t_n} \Delta(\phi_n(u))\mathrm{d}u\mathbf{1}_{t_n\le T}\right) - \dot{s}\left(\int_0^{T-t} \Delta(\phi(u))\mathrm{d}u\mathbf{1}_{t\le T}\right)\right| \\ &\le C\left|\int_0^{T-t_n} \Delta(\phi_n(u))\mathrm{d}u\mathbf{1}_{t_n\le T} - \int_0^{T-t} \Delta(\phi(u))\mathrm{d}u\mathbf{1}_{t\le T}\right| \end{aligned} \quad \text{(B.4)}$$

Assume that $t > T$, then for n large enough $t_n > T$ and the right-hand term of this inequality is zero. Assume to the contrary that $t \le T$. Since Δ is Lipschitzian on every compact of $(\dot{\pi}(\infty), \infty)$ and ϕ_n and ϕ take values in the same compact for $n \ge n_0$, there exists some constant $C' \ge 0$ such that we have for all $n \ge n_0$,

$$\begin{aligned}&\left|\int_0^{T-t_n} \Delta(\phi_n(u))\mathrm{d}u\mathbf{1}_{t_n\le T} - \int_0^{T-t} \Delta(\phi(u))\mathrm{d}u\mathbf{1}_{t\le T}\right| \\ &\quad\le \left|\int_0^{T-t_n} [\Delta(\phi_n(u)) - \Delta(\phi(u))]\mathrm{d}u\right| + \left|\int_{T-t_n}^{T-t} \Delta(\phi(u))\mathrm{d}u\right| \\ &\quad\le C'|T-t_n||\phi_n - \phi| + \left|\int_{T-t_n}^{T-t} \Delta(\phi(u))\mathrm{d}u\right|\end{aligned} \quad \text{(B.5)}$$

It is clear that the right-hand term of this last inequality converges to 0 as n tends to ∞. By (B.3)–(B.5) we have proved the continuity of F on Ω. Therefore we know from Theorem 20 that for every $\theta \in D$, since $(T,\theta) \in \Omega$, there exist $\gamma \in [T-\rho, T)$ and $z \in C((\gamma, T] + \rho; (\dot{\pi}(\infty), \infty))$ such that $\dot{z}(t) = F(t, z_t)$ for all $t \in (\gamma, T]$ and $z_T \equiv \theta$ on $[0,\rho]$.

In order to end the proof of Proposition 14 item 1, it remains to prove that z is valued in D and increasing on $(\gamma, T]$. The increasing feature of z is a direct consequence of the nonnegativity of $\dot{s}$ and the positivity of elements of $(\dot{\pi}(\infty), \infty)$. Besides, since $z(T) = z_T(0) = \theta \in D$, $D = (\dot{\pi}(\infty), \dot{\pi}(a)]$ and z is valued in $(\dot{\pi}(\infty), \infty)$, it is clear that it is in actual fact valued in D. This ends the proof of Proposition 14 item 1.

B.2. Proof of Proposition 14 item 2

Let (γ, z) be a solution of $\widetilde{BW}(\theta)$ with $\gamma > -\infty$. Since z is increasing and valued in D the limit $z(\gamma+) = \lim_{t\searrow\gamma} z(t)$ exists in $\bar{D} = [\dot{\pi}(\infty), \dot{\pi}(a)]$. Let us assume that $z(\gamma+) > \dot{\pi}(\infty)$. We have to prove that there exists some $\hat{\gamma} < \gamma$ and some $\hat{z} \in C((\hat{\gamma}, T]; D)$ such that $\hat{z}$ satisfies $\widetilde{BW}(\theta)$ on $(\hat{\gamma}, T]$ and such that $\hat{z}_{|(\gamma,T]} \equiv z$.

To do this, recall that z satisfies $z(t) = \theta - \int_t^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right)\right] \mathrm{d}u$ for all $t \in (\gamma, T]$, and hence, by passing to the limit as t goes to γ

$$z(\gamma+) = \theta - \int_\gamma^T \left[\lambda z(u) + \dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right)\right] \mathrm{d}u.$$

Therefore, it suffices to prove that

$$\begin{cases} \text{there exists some } \hat{\gamma} < \gamma \text{ and some } \tilde{z} \in C((\hat{\gamma}, \gamma]; D) \text{ such that} \\ \tilde{z}(t) = z(\gamma+) - \displaystyle\int_t^\gamma \left[\lambda \tilde{z}(u) + \dot{s}\left(\int_u^\gamma \Delta(\tilde{z}(v))\mathrm{d}v + \int_\gamma^T \Delta(z(v))\mathrm{d}v\right)\right] \mathrm{d}u, \\ \forall t \in (\hat{\gamma}, \gamma]. \end{cases} \tag{B.6}$$

Indeed, the result will then hold for $(\hat{\gamma}, \hat{z})$ where $\hat{z}$ is defined by setting $\hat{z}_{|(\hat{\gamma},\gamma]} \equiv \tilde{z}$ and $\hat{z}_{|(\gamma,T]} \equiv z$. In order to establish (B.6) one can use Theorem 20 for some $\rho > 0$, with $\Omega = (\gamma - \rho, T) \times C([0, \rho]; (\dot{\pi}(\infty), \infty))$ and F defined on Ω by

$$F(t, \phi) = \lambda\phi(0) + \dot{s}\left(\int_0^{\gamma - t} \Delta(\phi(u))\mathrm{d}u + \int_\gamma^T \Delta(z(u))\mathrm{d}u\right).$$

It can be checked that F is continuous on Ω. Since $(\eta, \theta) \triangleq (\gamma, z_\gamma)$ is in Ω, this provides the existence of some $\tilde{z} \in C((\hat{\gamma}, \gamma]; (\dot{\pi}(\infty), \infty))$ satisfying (B.6). It can be proved that $\tilde{z}$ is increasing and valued in D as in the proof of item 1. This ends the proof of Proposition 14 item 2.

B.3. Proof of Proposition 14 item 3

Let $\eta \in (-\infty, T]$ and let L be a compact subset of D. Consider (γ, z) a solution of $\widetilde{BW}(\theta)$ and (γ', z') a solution of $\widetilde{BW}(\theta')$. Let $t \in [\eta, T]$ with $t > \gamma, \gamma'$ and such that z and z' both map $[t, T]$ into L. For every $\sigma \in [t, T]$, we have

$$z(\sigma) - z'(\sigma) = \theta - \theta' - \int_\sigma^T \lambda[z(u) - z'(u)]\,\mathrm{d}u - \int_\sigma^T \left[\dot{s}\left(\int_u^T \Delta(z(v))\mathrm{d}v\right) - \dot{s}\left(\int_u^T \Delta(z'(v))\mathrm{d}v\right)\right] \mathrm{d}u$$

Since z and z' are increasing, for all $u \in [t, T]$, they both map $[u, T]$ into L. By continuity of Δ on D, this implies that, there exists some compact subset of $\mathbb{R}^+$ which contains the sets $\{\int_u^T \Delta(z(v))\mathrm{d}v\,,\ u \in [\eta, T]\}$ and $\{\int_u^T \Delta(z'(v))\mathrm{d}v\,,\ u \in [\eta, T]\}$. Therefore, since $\dot{s}$ is Lipschitzian on every compact subset of $\mathbb{R}^+$, there exists some constant $C_{\eta,L} \geq 0$ such that we have, for every $\sigma \in [t, T]$,

$$|z(\sigma) - z'(\sigma)| \leq |\theta - \theta'| + \int_\sigma^T \lambda|z(u) - z'(u)|\,\mathrm{d}u + C_{\eta,L}\int_\sigma^T \left|\int_u^T \Delta(z(v))\mathrm{d}v - \int_u^T \Delta(z'(v))\mathrm{d}v\right| \mathrm{d}u$$

Using now the Lipschitzianity on compacts of Δ, we obtain the existence of $C_{\eta,L} \geq 0$ such that we have, for every $\sigma \in [t, T]$,

$$|z(\sigma) - z'(\sigma)| \leq |\theta - \theta'| + \int_\sigma^T \lambda|z(u) - z'(u)|\mathrm{d}u + C_{\eta,L}\int_\sigma^T \int_u^T |z(v) - z'(v)|\,\mathrm{d}v\,\mathrm{d}u$$

This implies that, for all σ in $[t, T]$, we have

$$|z(\sigma) - z'(\sigma)| \leq |\theta - \theta' + (\lambda + C_{\eta,L}(T - \eta))\int_\sigma^T |z(u) - z'(u)|\mathrm{d}u \leq |\theta - \theta'| + C'_{\eta,L}\int_\sigma^T |z(u) - z'(u)|\mathrm{d}u.$$

By applying Gronwall's Lemma to the last inequality and since $\eta \leq \sigma$ we obtain

$$|z(\sigma) - z'(\sigma)| \leq |\theta - \theta'|\mathrm{e}^{C'_{\eta,L}(T-\sigma)} \leq |\theta - \theta'|\mathrm{e}^{C'_{\eta,L}(T-\eta)}.$$

This ends the proof of Proposition 14.

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