Live Fast, Die Young

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December 5, 2013

Abstract

Irrational agents are driven out of the market by the rational ones. Intuitively, this mechanism should favor learning : irrational agents observing that rational agents are being more successful should adopt the same beliefs as the most successful ones. In this note, we show that the threat of elimination is not sufficient to push the agents towards rationality: a shorter "life" might be more rewarding than a longer one. More precisely, in a model with rational and irrational agents, we show that there are situations where irrational agents might rationally stay irrational in the sense that their ex-ante and ex-post welfare levels over their whole life are higher than (1) the welfare level that they would reach if they adopted rational expectations, (2) the welfare level reached by the rational agents, (3) the welfare level that they would have if they suddenly had the opportunity to swap their optimal allocation against the optimal allocation of a rational agents.

"*Passionate lived, reasonable lasted*", Chamfort

In the classical financial economics theory, decision makers are assumed to have homogenous and rational expectations. This assumption has been

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the basis for many developments in finance like the portfolio selection model (Markowitz, 1952) and the CAPM (Sharpe 1964 and Lintner 1965).

Many authors have shown that agents who make inaccurate predictions are driven out of the market by those who are more accurate (see e.g. Sandroni, 2000). Intuitively, this should provide a robust justification for the rational expectations hypothesis based on a learning/evolutionary argument: irrational agents should see that rational agents are being more successful and they should adopt the same beliefs as the most successful ones. Yan (2010) provided a first limitation to this argument: even though irrational agents are eliminated, it might take hundreds of years to eliminate them.

In this note, we provide another limit to learning : the threat of elimination is not necessarily sufficient to push the agents towards rationality: a shorter "life" might be more rewarding than a longer one.

More precisely, we show, in a dynamic model with two groups of agents - rational and irrational ones -, that there are situations where the irrational group might rationally stay irrational in the sense that its ex-ante and ex-post welfare levels over the whole life is higher than

- the welfare level that it would reach if it adopted rational expectations,
- the average welfare level reached by the rational group,
- the welfare level that it would have if it suddenly had the opportunity to swap its optimal allocation against the optimal allocation of the rational group.

Even though strategic behavior is not part of our model, our results shed some light on a debate initiated by Grossman and Stiglitz (1980) around the following simple question : is it economically rational to acquire information or, in other words, is there an economic rationale for learning? They show that when acquiring information is costly (in their model in terms of money but it could also be in terms of efforts), the markets can not be informationally efficient. Kyle (1989) solves the paradox by introducing imperfect competition: since agents' demand schedules impact prices, agents should take this impact into account. In both models, equilibrium prices convey information about aggregate information in the economy and each agent learns from price observation through a Bayesian learning process. However, as underlined by Routledge (1999), all these rational expectations models where agents extract information from prices do not address how agents acquire sufficient knowledge about economic structure, parameters,... Furthermore, Bayesian learning leads to complex mechanisms where the learning process of each agent should be consistent with the learning process of all others. Kyle (1989) model shows that imperfect competition and Bayesian learning permit to each agent to extract a specific rent from his own information and acquiring information becomes useful while a free-riding behavior emerges in Grossman and Stiglitz (1980). However, both models take learning as an intrinsic element of agents' behavior. Therefore, the question remains open: is it rational for agents to spend effort on learning? In a dynamic model constructed as a succession of Grossman and Stiglitz models, Routledge (1999) shows that adaptative or evolutionary learning where agents imitate successful behavior permits to converge towards the rational anticipation equilibrium when the initial proportion of rational agents is sufficiently high. This result is obtained in a model where all agents have the same utility function. Learning and convergence to rational anticipations result then from a basic rational behavior (without assuming unbounded rationality nor a complete and perfect knowledge of the economy). However, when agents have heterogenous levels of risk aversion and in an otherwise similar framework (succession of static equilibria), Jouini et al (2013) have shown that adaptative learning leads to beliefs subjectivity and heterogeneity : the objective belief is not optimal, and agents differ in their optimal beliefs.

In the present note we adopt a truly dynamic framework: infinitely lived agents maximize their intertemporal welfare. In such a framework, our results above can then be interpreted as follows: it might occur that the most successful agents (for a large collection of criteria) are the irrational ones. They will then never learn through adaptative learning. In other words, irrational behaviors might rationally survive even if irrational agents regularly compare their performances with those of rational ones.

The paper is organized as follows. Section 2 describes the model, Section 3 provides a numerical illustration and Section 4 concludes. All proofs are in the Appendix.

1 The model

A filtered probability space $(\Omega, F, (F_t), P)$ describing uncertainty is given. We consider a continuous-time pure exchange Arrow-Debreu economy, with a single consumption good and two groups of agents that maximize their expected utility from future consumption. Both groups have exponential utility functions but differ in their levels of risk tolerance as well as in their subjective beliefs about the future of the economy. Moreover, the first group consists of rational agents while the second group consists of irrational agents, with the same common subjective belief Q equivalent to P and whose density process with respect to P is given by M.

A classical representative agent approach permits to aggregate all the agents in the first (resp. second) group into a unique agent denoted by Agent 1 (resp. Agent 2) whose belief is given by P (resp. Q), whose initial wealth is given by the total wealth within the group and whose risk tolerance level is the sum (average) of the risk tolerance levels across the group when dealing with finitely (infinitely) many agents.

In the next, we assume that wealth and beliefs are independent and both groups have then the same aggregated wealth and both representative agents have then the same initial endowment $\frac{1}{2}e^*$ where e^* denotes the aggregate endowment process in the economy. We make the assumption that the process e^* satisfies the following stochastic differential equations

$$de_t^* = \mu dt + \sigma dW_t$$

where W is a standard unidimensional $((F_t), P)$-Brownian motion and where μ and σ are given constants. From the point of view of the rational agents or equivalently of Agent 1, the total wealth is then a Brownian motion with drift parameter (mean by unit of time) $\mu_1 \equiv \mu$ and scale parameter σ (that corresponds to a variance by unit of time given by σ^2). We assume that, from the point of view of the irrational agents or of Agent 2, the total wealth is a Brownian motion with mean and variance by unit of time respectively given by $\mu_2 \equiv \mu + \delta_2 \sigma$ and σ^2 where δ_2 is a given constant. The parameter $\delta_2 = \frac{\mu_2 - \mu}{\sigma}$ then measures Agent 2 error in his perceived economic growth (normalized by the level of risk). For the symmetry, we introduce $\delta_1 = \frac{\mu_1 - \mu}{\sigma}$. Agent 2 is then optimistic if $\delta_2 > 0$ and pessimistic if $\delta_2 < 0$ whereas by construction, Agent 1 is rational and $\delta_1 = 0$. By Girsanov Theorem, Agent 2 belief corresponds to a density process (M_t) such that $M_0 = 1$ and $dM_t = \delta_2 M_t dW_t$.

Agent i has then a CARA utility function $u_i(c) = -\exp(-\frac{c}{\theta_i})$ where θ_i, $i = 1, 2$, represents the level of absolute risk tolerance of agent i (that is to say the sum or the average of the individual levels of risk tolerance across the agents of Group i). Agent 1 maximizes then

$$U_1(c) = E^P \left[\int_0^\infty - \exp\left(-\frac{c_t}{\theta_1} \right) dt \right]$$

whereas Agent 2 maximizes

$$U_2(c) = E^Q \left[\int_0^\infty - \exp\left(-\frac{c_t}{\theta_2} \right) dt \right].$$

As usual, an Arrow-Debreu equilibrium is defined by a positive density price process q^* and a pair of optimal consumption plans $(c_i^*)_{i=1,2}$ such that markets clear, i.e.,

$$\begin{cases} c_i^* = \arg\max \left\{ U_i(c) : E\left[\int_0^\infty q_t c_t dt \right] \leq \frac{1}{2} E\left[\int_0^\infty q_t e_t^* dt \right] \right\} \\ c_1^* + c_2^* = e^* \end{cases}.$$

In order to prove the existence of an equilibrium and to characterize it, we need the following condition.

Condition (C) $K \equiv \mu(\theta_1 + \theta_2) + \sigma \theta_2 \delta_2 + \frac{1}{2}\theta_1 \theta_2 \delta_2^2 - \frac{1}{2}\sigma^2 > 0.$

Condition (C) can be rewritten as follows: $\frac{\mu_1 \theta_1 + \mu_2 \theta_2}{\theta_1 + \theta_2}(\theta_1 + \theta_2) + \frac{1}{2}\theta_1 \theta_2 \delta_2^2 -$

$\frac{1}{2}\sigma^2 > 0$. In a rational economy, the usual condition for an equilibrium existence result is given by $\mu(\theta_1+\theta_2) - \frac{1}{2}\sigma^2$ which means that the total wealth is a desirable asset for the representative agent (whose absolute risk tolerance level is given by $\theta_1+\theta_2$). In Condition (C), the objective growth parameter μ is replaced by the average subjective one $\frac{\mu_1\theta_1+\mu_2\theta_2}{\theta_1+\theta_2}$. As seen in Jouini and Napp (2007), this risk tolerance weighted average belief corresponds to the representative[1] agent belief in such an heterogeneous beliefs framework. Assuming that the total wealth is a desirable asset for such a representative agent would give $\frac{\mu_1\theta_1+\mu_2\theta_2}{\theta_1+\theta_2}(\theta_1+\theta_2) - \frac{1}{2}\sigma^2 > 0$ which in turn implies Condition (C).

Proposition 1 *Under condition (C), there is only one equilibrium given by*

$$c^*_{i,t} = C_i t + D_i W_t + E_i,\ \ i = 1, 2$$

with $C_i = \theta_i\left(\frac{1}{2}v^2 - \frac{1}{2}\delta_i^2 + \frac{1}{\theta_1+\theta_2}\mu\right)$, $D_i = \theta_i\left(\delta_i - \delta + \frac{1}{\theta_1+\theta_2}\sigma\right)$, $E_i = \frac{1}{2}e_0^* + \frac{1}{2}\frac{F_i+G_iB}{\left(-A+\frac{1}{2}B^2\right)^2}$, $\delta = \frac{\theta_1\delta_1+\theta_2\delta_2}{\theta_1+\theta_2}$, $v = \sqrt{\frac{\theta_1\delta_1^2+\theta_2\delta_2^2}{\theta_1+\theta_2}}$, $A = \frac{1}{2}v^2 + \frac{\mu}{\theta_1+\theta_2}$, $B = \delta - \frac{\sigma}{\theta_1+\theta_2}$, $F_i = \frac{\theta_i-\theta_j}{\theta_1+\theta_2}\mu - \frac{\theta_2\theta_1}{\theta_1+\theta_2}\left(\delta_i^2 - \delta_j^2\right)$ *and* $G_i = \frac{\theta_i-\theta_j}{\theta_1+\theta_2}\sigma + 2\frac{\theta_2\theta_1}{\theta_1+\theta_2}(\delta_i - \delta_j)$ *for* $i = 1, 2$ *and* $j \neq i$.

Furthermore, unless agents are identical, i.e. $(\delta_1, \theta_1) = (\delta_2, \theta_2)$, *we have* $c_1^* \neq c_2^*$.

Remark In the rational setting, we have $c_i^* = \frac{\theta_i}{\theta_1+\theta_2}e^*$. Equation (1) shows how the optimal consumption plans in our framework deviate from the rational setting ones.

Remark Note that changing $(\mu, \sigma, \theta_1, \theta_2, e_0)$ by $(\alpha\mu, \alpha\sigma, \alpha\theta_1, \alpha\theta_2, \alpha e_0)$ changes (e^*, c_1^*, c_2^*) into $(\alpha e^*, \alpha c_1^*, \alpha c_2^*)$ and changing $(\mu, \sigma, \delta_2, t)$ by $\left(\alpha\mu, \sqrt{\alpha}\sigma, \sqrt{\alpha}\delta_2, \frac{t}{\alpha}\right)$ changes $\left(e_t^*, c_{1,t}^*, c_{2,t}^*\right)$ into $\left(e_{\alpha t}^*, c_{1,\alpha t}^*, \alpha c_{2,\alpha t}^*\right)$.

The following Proposition compares the asymptotic behavior of Agent 1 and Agent 2 optimal consumption plans in terms of instantaneous expected consumption $E[c_{i,t}]$ and in terms of $E[\exp(c_{i,t})]$. This last comparison makes sense if we consider our model as a log-linearization of a model with power

[1] Representative of the whole economy and not only of one of the two groups.

utility functions and where the total wealth follows a diffusion process with drift μ and volatility σ.

Proposition 2 *Under Condition (C), we have*

1. $\lim_{t\to\infty} \frac{E[c_{2,t}]}{E[c_{1,t}]} < 1$ *if and only if* $\theta_2 < \frac{\mu}{\mu - \theta_1 \delta_2^2} \theta_1$,

2. $\lim_{t\to\infty} \frac{E[\exp(c_{2,t})]}{E[\exp(c_{1,t})]} = 0$ *if and only if* $\theta_2 < \frac{2\mu+\sigma^2}{2\mu+\sigma^2+2\delta_2\theta_1(\sigma-\delta_2)} \theta_1$.

In particular, the first inequality holds for $\theta_2 < \theta_1$ and the second one holds for $\theta_2 < \theta_1$ and $\delta_2 > \sigma$ which means that when Agent 2 is more risk-averse he has, asymptotically and in average, a lower share of the total consumption which corresponds to the standard situation in the rational setting. Furthermore, when Agent 2 is optimistic enough, the expected value of $\exp(c_{2,t})$ is asymptotically dominated (the ratio goes to zero) by $\exp(c_{1,t})$.

Let us now compare agents welfare levels at the equilibrium.

Recall first that Agent 1 (Agent 2) represents the group of rational (irrational) agents. The next Proposition provides the link between $U_i(c_i^*)$ the equilibrium welfare level of Agent i and the individual equilibrium welfare levels withing Group i.

Proposition 3 *Agent i equilibrium welfare level is the risk tolerance weighted geometric average of the individual welfare levels within Group i.*

Therefore, comparing $U_1(c_1^*)$ and $U_2(c_2^*)$ amounts to comparing the average utility levels respectively among the rational and the irrational groups of agents.

Claim 4 *Under Condition (C), Agent i equilibrium welfare level $U_i(c_i^*)$ is given by*

$$U_i(c_i^*) = -\frac{\exp\left(-\frac{E_i}{\theta_i}\right)}{\left(\frac{C_i}{\theta_i} + \frac{1}{2}\delta_i^2\right) - \frac{1}{2}\left(\delta_i - \frac{D_i}{\theta_i}\right)^2}.$$

In particular, if Agent 2 is optimistic enough ($\delta_2 > 0$ and large enough) and more risk averse or less risk tolerant ($\theta_2 < \theta_1$) then $U_2(c_2^*) > U_1(c_1^*)$, the *welfare level of the optimistic agent is higher than the welfare level of the rational one.*

Rephrased in terms of groups and of individual agents, this means that when the irrationals are optimistic and when the total level of risk tolerance within the group of irrationals is smaller than the total level of risk tolerance within the group of rationals, then the average welfare level among the group of irrational agents is higher than the average welfare level among the group of rational agents.

We might also be interested by the ex-post welfare levels of both Agent 1 and Agent 2 (or within both groups). Indeed, even though Agent 2 believes that the probability of occurrence of the different states of the world is given by Q, the states of the world he will face during his life are governed by the objective probability and his ex-post welfare should then be measured under the objective probability. Obviously, there is no difference between ex-ante and ex-post welfare for Agent 1.

We denote then by $U_i^{\text{ex-post}}(c_i^*)$ the ex-post utility levels and we have

$$\begin{aligned} U_i^{\text{ex-post}}(c_i^*) &= E^P\left[\int_0^\infty -\exp\left(-\frac{c_{i,t}^*}{\theta_i}\right)dt\right], \\ U_1^{\text{ex-post}}(c_1^*) &= U_1(c_1^*). \end{aligned}$$

In order to compute these quantities explicitly we need the following additional condition

Condition (C1) $K_1 \equiv \mu(\theta_1+\theta_2) - \sigma\theta_1\delta_2 - \frac{1}{2}\sigma^2 - \frac{1}{2}\theta_1\theta_2\delta_2^2 - \theta_1^2\delta_2^2 > 0.$

We have then the following result.

Claim 5 *Under Conditions (C) and (C1), Agent i ex-post welfare level is given by*

$$U_i^{ex\text{-}post}(c_i^*) = -\exp\left(-\frac{E_i}{\theta_i}\right)\frac{1}{\left(\frac{C_i}{\theta_i} - \frac{1}{2}\frac{D_i^2}{\theta_i^2}\right)}.$$

Agent 2 observes agent 1 and the evolution of his welfare and of his wealth. When $U_2(c_2^*) > U_1(c_1^*)$ and $U_2^{\text{ex-post}}(c_2^*) > U_1^{\text{ex-post}}(c_1^*)$, there is no incentive for him to adapt his beliefs to those of Agent 1 when comparing the welfare levels. However, Agent 1 could also evaluate the welfare level he would reach with the consumption plan of Agent 1. Even though it is not clear how (with which beliefs) he could obtain that consumption plan at the equilibrium, it is quite natural for him to compare his own consumption plan with the consumption plan of the other agent and to dream about how much he would feel happy with this alternative consumption plan.

Let us introduce the following additional conditions.

Condition (C2) $K_2 \equiv \mu - \frac{1}{2}\frac{\theta_1}{\theta_2}\frac{\sigma^2}{\theta_1+\theta_2} + \frac{(2\theta_1+\theta_2)}{\theta_1+\theta_2}\sigma\delta_2 - \frac{1}{2}\frac{\theta_2+2\theta_1}{\theta_1+\theta_2}\theta_2\delta_2^2 > 0.$

Condition (C3) $K_3 \equiv \mu - \frac{1}{2}\frac{\theta_1}{\theta_2}\frac{\sigma^2}{\theta_1+\theta_2} + \sigma\frac{\theta_1}{\theta_1+\theta_2}\delta_2 + \frac{1}{2}\frac{\theta_2^2}{\theta_1+\theta_2}\delta_2^2 > 0.$

We have then the following result.

Claim 6 *Under Condition (C2), we have*

$$U_2(c_1^*) = -\frac{\exp\left(-\frac{E_1}{\theta_2}\right)}{\frac{C_1}{\theta_2} - \frac{1}{2}\frac{D_1^2}{\theta_2^2} + \delta_2\frac{D_1}{\theta_2}}.$$

If we further assume $(\delta_1, \theta_1) \neq (\delta_2, \theta_2)$ *and that Condition (C) is satisfied, we have*

$$U_2(c_1^*) < U_2(c_2^*)$$

and the irrational agent has a higher ex-ante welfare with his own plan rather than with Agent 1 one.

Under Condition (C3), we have

$$U_2^{\textit{ex-post}}(c_1^*) = -\frac{\exp\left(-\frac{E_1}{\theta_2}\right)}{\frac{C_1}{\theta_2} - \frac{1}{2}\frac{D_1^2}{\theta_2^2}}.$$

Now, let us determine the optimal allocation $\widetilde{c}_2^*$ when Agent 2 is rational. Since c_1^* and c_2^* are observable at the equilibrium, all the welfare levels we evaluated above, namely $U_1(c_1^*)$, $U_2(c_2^*)$, $U_1^{\text{ex-post}}(c_1^*)$, $U_2^{\text{ex-post}}(c_2^*)$ and $U_2(c_1^*)$, can be easily evaluated ex-ante or ex-post by both agents. On the contrary, $\widetilde{c}_2^*$ is not observable and would be computed only by an agent who is aware of his irrationality and would want to compare his welfare level when staying irrational with his welfare level when being rational or, in our framework where Agent 2 represents all the irrational agents, when all the irrational agents become rational. Furthermore, this computation is possible only when the agent under consideration knows all the caracterstics of the economy. The comparison of $U_2(c_2^*)$ and $U_2^{\text{ex-post}}(c_2^*)$ respectively with $U_2(\widetilde{c}_2^*)$ and $U_2^{\text{ex-post}}(\widetilde{c}_2^*)$ is then less natural from a learning point of view. It is however interesting from a theoretical point of view.

To make these comparisons, we need the two following additional conditions. Condition (C4) corresponds to Condition (C) for $\delta_2 = 0$ and Condition (C5) is weaker than Condition (C) when $\delta_2 < 0$.

Condition (C4) $K_4 \equiv \mu\,(\theta_1 + \theta_2) - \frac{1}{2}\sigma^2 > 0$.

Condition (C5) $K_5 \equiv \mu - \frac{1}{2}\frac{\sigma^2}{\theta_1+\theta_2} + \sigma\delta_2 > 0$.

Claim 7 *Under Conditions (C4) and (C5), the ex-ante and ex-post welfare levels for Agent 2 at the rational equilibrium consumption plan $\widetilde{c}_2^*$ are given by*

$$\begin{aligned} U_2(\widetilde{c}_2^*) &= -\frac{\exp\left(-\frac{E_0}{\theta_2}\right)}{\frac{C_0}{\theta_2} - \frac{1}{2}\frac{D_0^2}{\theta_2^2} + \delta_2\frac{D_0}{\theta_2}}, \\ U_2^{\text{ex-post}}(\widetilde{c}_2^*) &= -\frac{\exp\left(-\frac{E_0}{\theta_2}\right)}{\frac{C_0}{\theta_2} - \frac{1}{2}\frac{D_0^2}{\theta_2^2}}, \end{aligned}$$

with $C_0 = \frac{\theta_2}{\theta_1+\theta_2}\mu$, $D_0 = \frac{\theta_2}{\theta_1+\theta_2}\sigma$ *and* $E_0 = \frac{1}{2}e_0^* + \frac{1}{2}\frac{\left(\mu\theta_1+\mu\theta_2-\sigma^2\right)(\theta_1-\theta_2)}{\left(\mu\theta_1+\mu\theta_2-\frac{1}{2}\sigma^2\right)}$.

We are now in position to analyze specific examples.

2 A Numerical Example

Let us consider an economy with two categories of agents : the rational ones and the optimistic ones. We assume that the two groups of agents are equally sized. The wealth per capita in the whole economy is \$50,000 and we assume that there is no difference between the 2 groups in terms of wealth distribution. However, the agents in the first group are assumed to be less risk averse. In the first group, the average level of relative risk aversion is equal to $a_1 = 1.7$ while it is equal to $a_2 = 3.6$ in the second group. The objective growth level in the economy is equal to $\tilde{\mu} = 4.2\%$ with a volatility $\tilde{\sigma} = 3\%$. These objective parameters characterize the beliefs of the rational agents (Group 1) while the agents in Group 2 believe that the growth rate is equal to 4.8% (with a 3% volatility). We assume that all agents have CARA utility functions.

A reduced form of this economy consists in an economy with 2 agents endowed with CARA utility functions. Both agents have a $50,000 initial endowment and Agent 1 has an absolute risk tolerance level given by $\theta_1 = 29,412$ while Agent 2 has an absolute level of risk tolerance given by $\theta_2 = 13,889$. With the notations of our arithmetic growth model, we have $\mu = 4,200$, $\sigma = 3000$ and $\delta_2 = 0.2$.

With these reasonable parameters, we obtain $K = 1.9 \times 10^8$, $K_1 = 1.2 \times 10^8$, $K_2 = 4,521$, $K_3 = 4,477$, $K_4 = 4,096$, $K_5 = 4,696$, and they are all positive. Then Conditions (C), (C1), (C2), (C3) and (C4) are all satisfied and we have $U_1(c_1^*) = -2.41$, $U_2(c_2^*) = -0.137$, $U_1^{\text{ex-post}}(c_1^*) = -2.41$, $U_2^{\text{ex-post}}(c_2^*) = -0.226$, $U_2^{\text{ex-post}}(c_1^*) = -0.242$, $U_2(c_1^*) = -0.239$ and $U_2(\widetilde{c}_2^*) = -0.146$ and $U_2^{\text{ex-post}}(\widetilde{c}_2^*) = -0.168$ which gives

$$U_2(c_2^*) > U_2(\widetilde{c}_2^*) > U_2^{\text{ex-post}}(\widetilde{c}_2^*) > U_2^{\text{ex-post}}(c_2^*) > U_2(c_1^*) > U_1(c_1^*) = U_1^{\text{ex-post}}(c_1^*) > U_2^{\text{ex-post}}(c_1^*).$$

We have then

- $U_2(c_2^*) > U_1(c_1^*)$, the ex-ante (subjective) welfare of Agent 2 is higher than the ex-ante (objective/subjective) welfare of Agent 1,
- $U_2^{\text{ex-post}}(c_2^*) > U_1^{\text{ex-post}}(c_1^*)$, the ex-post (objective) welfare of Agent 2 is higher than the ex-ante/ex-post welfare of Agent 1,
- $U_2(c_2^*) > U_2(\widetilde{c}_2^*)$, the ex-ante welfare level of Agent 2 is higher than

the welfare level he would reach in a model where all agents would be rational,

- $U_2(c_2^*) > U_2(c_1^*)$ and $U_2^{\text{ex-post}}(c_2^*) > U_2^{\text{ex-post}}(c_1^*)$, both ex-ante and ex-post welfare levels of Agent 2 are higher than those that he would reach if endowed with the consumption plan of Agent 1.

The only thing we don't have is $U_2^{\text{ex-post}}(c_2^*) > U_2^{\text{ex-post}}(\widetilde{c}_2^*)$. The irrational agent would reach a higher ex-post welfare level in a fully rational framework.

As underlined above, since $\widetilde{c}_2^*$ is not observable, the comparison of $U_2(c_2^*)$ and $U_2^{\text{ex-post}}(c_2^*)$ respectively with $U_2(\widetilde{c}_2^*)$ and $U_2^{\text{ex-post}}(\widetilde{c}_2^*)$ is less natural from a learning point of view. However, from a theoretical point of view, it would be interesting to know if there are situations where all the inequalities above are satisfied and where we further have $U_2^{\text{ex-post}}(c_2^*) > U_2^{\text{ex-post}}(\widetilde{c}_2^*)$. Such a situation is given with the following parameters : $e_0 = 0$, $\mu = 1$, $\sigma = 1$, $\theta_1 = 1$, $\theta_2 = 0.5$ and $\delta_2 = 0.05$. With these parameters, we have $(K, K_1, K_2, K_3, K_4, K_5) = (1.0, 0.9, 0.4, 0.4, 0.7, 0.7)$ and then Conditions (C) to (C4) are all satisfied. Furthermore, we have $U_1(c_1^*) = -2.556$, $U_2(c_2^*) = -1.616$, $U_1^{\text{ex-post}}(c_1^*) = -2.556$, $U_2^{\text{ex-post}}(c_2^*) = -1.751$, $U_2(c_1^*) = -2.449$, $U_2^{\text{ex-post}}(c_1^*) = -2.774$, $U_2(\widetilde{c}_2^*) = -1.630$ and $U_2^{\text{ex-post}}(\widetilde{c}_2^*) = -1.752$ which gives

$$U_2(c_2^*) > U_2(\widetilde{c}_2^*) > U_2^{\text{ex-post}}(c_2^*) > U_2^{\text{ex-post}}(\widetilde{c}_2^*) > U_2(c_1^*) > U_2^{\text{ex-post}}(c_1^*) > U_1(c_1^*) = U_1^{\text{ex-post}}(c_1^*).$$

This result can be enlightened by the analysis conducted in Jouini et al (2013) where it is shown that a strategic behavior leads to beliefs subjectivity and heterogeneity and to a positive correlation between pessimism and risk tolerance. The intuition is as follows. For a very risk averse agent, his demand in the risky asset is negative, so that his expected utility from trade is increasing in the price of the risky asset. The choice of an optimistic belief is associated to a higher demand, hence to a higher price, and the optimal belief balances this benefit of optimism against the costs of worse decision making. In particular, this means that, in a 2 agents (or group of agents) model, rationality is not the best response to rationality. In particular, in a model where beliefs are exogenously given as in the current paper, if the irrational agents are both more risk averse and more optimistic than the rational ones, then they might reach a higher utility level than if they were rational. This is exactly what happens in our dynamic example and, as underlined above, we obtain much more than that since it appears that irrationality dominates rationality for many other possible criteria.

3 Conclusion

In this note, we provided an example where

- threat of elimination is not sufficient to push irrational agents towards rationality and
- rational agents performances are not sufficiently high to generate learning through an adaptative process: imitating successful behavior does not lead to rationality.

There are then situations where there is no economic rationale for learning and research of truth. In our example, as in Plato's cave, rational agents who know the truth (true probability) do not deviate from their beliefs but there is no (economically rational) way to transmit their knowledge to the irrational agents.

To be more precise and to pursue with Plato's cave analogy, the ways and processes we consider are based on observable things only : each agent (or group of agents) compares his allocation and his welfare with the allocation and the welfare of the other agent (or group of agents). They are unable to compare their welfare with the welfare they would reach in other possible configurations of the whole economy (different proportions of rationals and irrationals, new equilibrium prices if all/some agents in a given group suddenly switch to another belief,...).

Evolutionary mechanism based on natural selection (adaptative learning) but also on random variations (genetic variations) would perhaps permit to explore such new configurations and converge to rationality. Similarly, we might have better convergence properties for mechanisms where agents meet randomly by pairs and where, in a first step, each agent in the pair adopts the behavior of the other agent during a given evaluation period deciding if he maintains this new belief or if he goes back to the old one. Indeed, in such a mechanism, each agent, would have, at a given moment, the possibility to compare his welfare level when being rational and his welfare level when being irrational, the proportion of rationals and irrationals being momentarily constant. In a model with a large number of agents, each agent would then have the possibility to benefit from others' optimism without paying the cost of worse decision making.

Such random variations and random switches might be compared to randomly freeing prisoners from the cave. Does this mean that only observation of truth permits learning truth and that truth is not accessible through (economically) rational thinking/analysis?

References

[1] Grossman, S. and J. Stiglitz, 1980. On the Impossibility of Informationally Efficient Markets, American Economic Review, 70, 393-408.

[2] Jouini, E. and C. Napp, 2007. Consensus consumer and intertemporal asset pricing under heterogeneous beliefs, Review of Economic Studies, 74, 1149-1174.

[3] Jouini, E., Napp, C. and Y. Viossat, 2013. Evolutionary Beliefs and Financial Markets, Review of Finance, 2013, 17, 727-766.

[4] Kyle, A., 1989. Informed Speculation with Imperfect Competition, Review of Economic Studies, 56, 317-355.

[5] Lintner, J., 1965. The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets. Review of Economics and Statistics, 47, 13-37.

[6] Markowitz, H., 1952. Portfolio selection. Journal of Finance, 7, 77-91.

[7] Routledge, B., 1999. Adaptive Learning in Financial Markets. Review of Financial Studies, 12, 1165-1202.

[8] Sandroni, A., 2000. Do markets favor agents able to make accurate prediction. Econometrica, 68, 1303-1341.

[9] Sharpe, W.F., 1964. Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk. Journal of Finance, 19, 425-442.

[10] Yan, H., 2010. Is Noise Trading Cancelled out by Aggregation ? Management Science, 56, 1047-1059.

Appendix

Proof of Proposition 1. Let us solve for an equilibrium. The first order conditions for Agent 1 and Agent 2 are given by

$$M_i \exp\left(-\frac{c_i^*}{\theta_i}\right) = \lambda_i q^*,\ \ i = 1, 2,$$

where $M_2 = M$, where $M_1 = 1$ is introduced for the symmetry in the formulas and where λ_1 and λ_2 are Lagrange multipliers and are such that the budget constraint is saturated for each agent, i.e. $E\left[\int_0^T q_t^* c_{i,t}^* dt\right] = \frac{1}{2} E\left[\int_0^T q_t^* e_t^* dt\right]$, $i = 1, 2$.

This gives

$$c_i^* = \theta_i \ln \frac{M_i}{\lambda_i q^*},\ \ i = 1, 2,$$

and since we have $c_1 + c_2 = e^*$ we obtain

$$q^* = \frac{1}{\lambda_1^{\frac{\theta_1}{\theta_1+\theta_2}} \lambda_2^{\frac{\theta_2}{\theta_1+\theta_2}}} M_1^{\frac{\theta_1}{\theta_1+\theta_2}} M_2^{\frac{\theta_2}{\theta_1+\theta_2}} \exp\left(-\frac{e^*}{\theta_1+\theta_2}\right)$$

and from there

$$\begin{aligned} c_{i,t}^* &= \frac{\theta_i}{\theta_1+\theta_2} e^* + \frac{\theta_1\theta_2}{\theta_1+\theta_2} \ln\left(\frac{M_i}{M_j}\frac{\lambda_j}{\lambda_i}\right),\ \ i = 1, 2,\ \ j \neq i, \qquad (1) \\ &= C_i t + D_i W_t + E_i \end{aligned}$$

where C_i and D_i are as above and where $E_i = \frac{\theta_i}{\theta_1+\theta_2} e_0^* + \frac{\theta_1\theta_2}{\theta_1+\theta_2} \ln \frac{\lambda_j}{\lambda_i}$.

It only remains to determine the values of λ_1 and λ_2 or to directly determine the values of E_1 and E_2. Since the agents have the same initial wealth, we should have

$$E\left[\int_0^\infty q_t^* c_{1,t}^* dt\right] = E\left[\int_0^\infty q_t^* c_{2,t}^* dt\right]. \tag{2}$$

We have

$$\begin{aligned}
& \lambda_1^{\frac{\theta_1}{\theta_1+\theta_2}} \lambda_2^{\frac{\theta_2}{\theta_1+\theta_2}} \exp\left(\frac{e_0^*}{\theta_1+\theta_2}\right) E\left[\int_0^\infty q^* c_{i,t}^* dt\right] \\
= & \int_0^\infty E\left[M_1^{\frac{\theta_1}{\theta_1+\theta_2}} M_2^{\frac{\theta_2}{\theta_1+\theta_2}} \exp\left(-\frac{e_t^* - e_0^*}{\theta_1+\theta_2}\right) c_{i,t}^*\right] dt \\
= & \int_0^\infty E\left[\exp(-\frac{1}{2}v^2 t + \delta W) \exp\left(-\frac{\mu t + \sigma W_t}{\theta_1+\theta_2}\right) (C_i t + D_i W_t + E_i)\right] dt.
\end{aligned}$$

Simple computations give

$$\begin{aligned}
& E\left[\exp(-\frac{1}{2}v^2 t + \delta W_t) \exp\left(-\frac{\mu t + \sigma W_t}{\theta_1+\theta_2}\right) (C_i t + D_i W_t + E_i)\right] \\
= & (C_i t + D_i B t + E_i) \exp\left((-A + \frac{1}{2}B^2) t\right) \qquad (3)
\end{aligned}$$

with $A = \frac{1}{2}v^2 + \frac{\mu}{\theta_1+\theta_2}$ and $B = \delta - \frac{\sigma}{\theta_1+\theta_2}$.

Condition (C) gives us $-A + \frac{1}{2}B^2 < 0$ and we may integrate (3) from 0 to ∞ and we have

$$E\left[\int_0^\infty M_1^{\frac{\theta_1}{\theta_1+\theta_2}} M_2^{\frac{\theta_2}{\theta_1+\theta_2}} \exp\left(-\frac{e_t^*}{\theta_1+\theta_2}\right) c_{i,t}^* dt\right] = -\frac{E_i}{-A + \frac{1}{2}B^2} + \frac{C_i + D_i B}{\left(-A + \frac{1}{2}B^2\right)^2}$$

Equation (2) gives then

$$E_1 - E_2 = \frac{F_1 + G_1 B}{\left(-A + \frac{1}{2}B^2\right)}$$

where $F_1 = C_1 - C_2 = \frac{\theta_1-\theta_2}{\theta_1+\theta_2}\mu - \frac{\theta_2\theta_1}{\theta_1+\theta_2}\left(\delta_1^2 - \delta_2^2\right)$ and $G_1 = D_1 - D_2 = \frac{\theta_1-\theta_2}{\theta_1+\theta_2}\sigma + 2\frac{\theta_2\theta_1}{\theta_1+\theta_2}\left(\delta_1 - \delta_2\right)$. From the expressions of E_1 and E_2 we also have

$$E_1 + E_2 = e_0^*$$

which gives

$$E_i = \frac{1}{2}\frac{F_i + G_i B}{\left(-A + \frac{1}{2}B^2\right)} + \frac{1}{2}e_0^*.$$

Let us now solve for $c_1^* = c_2^*$ or equivalently for $C_1 = C_2$, $D_1 = D_2$ and $E_1 = E_2 = \frac{1}{2}e_0^*$. This immediately leads to $(\delta_2, \theta_2) = (\delta_1, \theta_1)$ or to $(\delta_2, \theta_2) = \left(-\frac{2\mu}{\sigma}, \frac{\sigma^2\theta_1}{\sigma^2-4\mu\theta_2}\right)$. For the second solution, Condition (C) becomes $\sigma^2 < 4\mu\theta_2 < -\sigma^2$ which permits to conclude.

Proof of Proposition 2. It is immediate that

$$\frac{E\left[c_2\right]}{E\left[c_1\right]} = \frac{C_2 t + E_2}{C_1 t + E_1} \rightarrow \frac{C_2}{C_1} = \frac{\theta_2}{\theta_1}\frac{\mu - \frac{1}{2}\theta_1\delta_2^2}{\mu + \frac{1}{2}\theta_2\delta_2^2}$$

which is smaller than 1 for $\theta_2 < \frac{\mu\theta_1}{\mu-\theta_1\delta_2^2}$. Similarly, we have

$$\begin{aligned}\frac{E\left[\exp(c_{2,t})\right]}{E\left[\exp(c_{1,t})\right]} &= \frac{E\left[\exp\left(C_2 t + D_2 W_t + E_2\right)\right]}{E\left[\exp\left(C_1 t + D_1 W_t + E_1\right)\right]}, \\ &= \exp(E_2 - E_1)\frac{\exp\left(C_2 + \frac{1}{2}D_2^2\right)t}{\exp\left(C_1 + \frac{1}{2}D_1^2\right)t}.\end{aligned}$$

The term within the exponential is equal to $-\frac{1}{2}\frac{2\mu\theta_1-2\mu\theta_2-2\sigma\theta_1\theta_2\delta_2+\sigma^2\theta_1-\sigma^2\theta_2+2\theta_1\theta_2\delta_2^2}{\theta_1+\theta_2}t$. It is clearly negative for $\theta_2 < \frac{2\mu\theta_1+\sigma^2\theta_1}{2\mu+\sigma^2+2\delta_2\theta_1(\sigma-\delta_2)}$.

Proof of Proposition 3. Let us index by $\gamma \in \Gamma_i$ the individual members of Group i and let us examine more in detail how Agent i optimal consumption c_i^* is distributed among the members of Group i. For a given $\gamma \in \Gamma_i$, we denote by $c_{i,\gamma}^*$, $\theta_{i,\gamma}$ and $w_{i,\gamma}$ respectively the optimal consumption, the risk tolerance level and the wealth share within Group i of γ. By construction, we have $\sum_{\gamma\in\Gamma_i} c_{i,\gamma}^* = c_\gamma^*$, $\sum_{\gamma\in\Gamma_i}\theta_{i,\gamma} = \theta_i$ and $\sum_{\gamma\in\Gamma_i} w_{i,\gamma} = 1$. It is easy to check that, at the equilibrium, we have $c_{i,\gamma}^* = \frac{\theta_{i,\gamma}}{\theta_i}c_{i,\gamma}^* + (w_{i,\gamma} - \frac{\theta_{i,\gamma}}{\theta_i})q \cdot c_i^*$ where

$q \cdot c_i^* = E\left[\int_0^\infty q_t c_{i,t}^* dt\right]$. We have then

$$\begin{aligned} U_\gamma\left(c_{i,\gamma}^*\right) &= E\left[\int_0^\infty -\exp\left(-\frac{c_{i,\gamma,t}^*}{\theta_{i,\gamma}}\right)dt\right] \\ &= U_i\left(c_i^*\right)\exp\left(-\frac{q\cdot c_i^*}{\theta_i}\right)\exp\left(\frac{w_{i,\gamma}}{\theta_{i,\gamma}}q\cdot c_i^*\right) \end{aligned}$$

and $\prod_{\gamma\in\Gamma_i} U_\gamma\left(c_{i,\gamma}^*\right)^{\frac{\theta_{i,\gamma}}{\theta_i}} = U_i\left(c_i^*\right)$.

Proof of Claim 4. From the first order conditions we have

$$\frac{1}{\lambda_1}M_1\exp\left(-\frac{c_1^*}{\theta_i}\right) = \frac{1}{\lambda_2}M_2\exp\left(-\frac{c_2^*}{\theta_i}\right)$$

hence

$$\frac{1}{\lambda_1}U_1\left(c_1^*\right) = \frac{1}{\lambda_2}U_2\left(c_2^*\right)$$

and

$$\frac{U_2\left(c_2^*\right)}{U_1\left(c_1^*\right)} = \frac{\lambda_2}{\lambda_1}.$$

We know that $E_i = \frac{\theta_i}{\theta_1+\theta_2}e_0^* + \frac{\theta_1\theta_2}{\theta_1+\theta_2}\ln\frac{\lambda_j}{\lambda_i}$ which gives

$$\frac{U_2\left(c_2^*\right)}{U_1\left(c_1^*\right)} = \exp\left(\frac{\theta_1+\theta_2}{\theta_1\theta_2}E_1 - \frac{1}{\theta_2}e_0^*\right).$$

A direct computation gives

$$\begin{aligned} U_i\left(c_i^*\right) &= E\left[\int_0^\infty -\exp\left(\delta_i W_t - \frac{1}{2}\delta_i^2 t\right)\exp\left(-\frac{C_i t + D_i W_t + E_i}{\theta_i}\right)dt\right] \\ &= -\exp\left(-\frac{E_i}{\theta_i}\right)\int_0^\infty \exp\left(-\left(\frac{C_i}{\theta_i}+\frac{1}{2}\delta_i^2\right)t + \frac{1}{2}\left(\delta_i - \frac{D_i}{\theta_i}\right)^2 t\right)dt \\ &= -\frac{\exp\left(-\frac{E_i}{\theta_i}\right)}{\left(\frac{C_i}{\theta_i}+\frac{1}{2}\delta_i^2\right) - \frac{1}{2}\left(\delta_i - \frac{D_i}{\theta_i}\right)^2} \end{aligned}$$

where the convergence of the last integral is insured by Condition (C).

Since all the welfare functions are negative, we have $U_2(c_2^*) > U_1(c_1^*)$ if and only if $\frac{U_2(c_2^*)}{U_1(c_1^*)} < 1$ or if

$$\frac{1}{2}\frac{\theta_1+\theta_2}{\theta_1\theta_2}e_0^* - \frac{1}{\theta_2}e_0^* - \frac{1}{2}\frac{(\theta_1+\theta_2)^2}{\theta_1\theta_2}\frac{(\theta_1-\theta_2)\left(\mu - \frac{\sigma^2}{\theta_1+\theta_2} + \frac{\theta_1\theta_2}{\theta_1+\theta_2}\delta_2^2\right) + \frac{\theta_2}{\theta_1+\theta_2}(3\theta_1-\theta_2)\sigma\delta_2}{\mu(\theta_1+\theta_2) + \sigma\theta_2\delta_2 + \frac{1}{2}\theta_1\theta_2\delta_2^2 - \frac{1}{2}\sigma^2}.$$

If $\theta_2 < \theta_1$, we have $\frac{1}{2}\frac{\theta_1+\theta_2}{\theta_1\theta_2}e_0^* - \frac{1}{\theta_2}e_0^* < 0$. Furthermore, under Condition (C), the last term is negative if and only if $(\theta_1-\theta_2)\left(\mu - \frac{\sigma^2}{\theta_1+\theta_2} + \frac{\theta_1\theta_2}{\theta_1+\theta_2}\delta_2^2\right) + \frac{\theta_2}{\theta_1+\theta_2}(3\theta_1-\theta_2)\sigma\delta_2$ is positive which is the case for $\theta_2 < \theta_1$ and $\delta_2 > 0$ and large enough.

Proof of Claim 5. We have

$$\begin{aligned} U_i^{\text{ex-post}}(c_i^*) &= \int_0^\infty -\exp\left(-\frac{C_i t + E_i}{\theta_i}\right) E^P\left[\exp\left(-\frac{D_i W_t}{\theta_i}\right)\right] dt \\ &= -\exp\left(-\frac{E_i}{\theta_i}\right)\int_0^\infty \exp\left(-\frac{C_i}{\theta_i}t\right) E^P\left[\exp\left(-\frac{D_i W_t}{\theta_i}\right)\right] dt \\ &= -\exp\left(-\frac{E_i}{\theta_i}\right)\int_0^\infty \exp\left(-\left(\frac{C_i}{\theta_i} - \frac{1}{2}\frac{D_i^2}{\theta_i^2}\right)t\right) dt \\ &= -\exp\left(-\frac{E_i}{\theta_i}\right)\frac{1}{\left(\frac{C_i}{\theta_i} - \frac{1}{2}\frac{D_i^2}{\theta_i^2}\right)} \end{aligned}$$

as far as $\frac{C_i}{\theta_i} - \frac{1}{2}\frac{D_i^2}{\theta_i^2} > 0$ or equivalently

$$\begin{aligned} \mu(\theta_1+\theta_2) + \sigma\theta_2\delta_2 - \frac{1}{2}\sigma^2 + \frac{1}{2}\theta_1\theta_2\delta_2^2 &> 0, \\ \mu(\theta_1+\theta_2) - \sigma\theta_1\delta_2 - \frac{1}{2}\sigma^2 - \frac{1}{2}\theta_1\theta_2\delta_2^2 - \theta_1^2\delta_2^2 &> 0. \end{aligned}$$

These conditions respectively correspond to Conditions (C) and (C1).

Proof of Claim 6. We have

$$
\begin{aligned}
U_2(c_1^*) &= \int_0^\infty -\exp\left(-\frac{C_1 t + E_1}{\theta_2}\right)\exp(-\frac{1}{2}\delta_2^2 t)E^P\left[\exp\left(\left(\delta_2 - \frac{D_1}{\theta_2}\right)W_t\right)\right]dt \\
&= -\exp\left(-\frac{E_1}{\theta_2}\right)\int_0^\infty \exp\left((-\frac{C_1}{\theta_2} + \frac{1}{2}\frac{D_1^2}{\theta_2^2} - \delta_2\frac{D_1}{\theta_2})t\right)dt
\end{aligned}
$$

which converges under Condition (C2) and leads to

$$
U_2(c_1^*) = -\frac{\exp\left(-\frac{E_1}{\theta_2}\right)}{\frac{C_1}{\theta_2} - \frac{1}{2}\frac{D_1^2}{\theta_2^2} + \delta_2\frac{D_1}{\theta_2}}.
$$

Since both agents have the same initial endowment, c_1^* satisfies then the budget constraint of Agent 2. By optimality, we have $U_2(c_1^*) \leq U_2(c_2^*)$. We even have $U_2(c_2^*) > U_2(c_1^*)$ as far as $c_2^* \neq c_1^*$ which is the case, as seen above, as far as the agents are not identical $(\delta_1, \theta_1) \neq (\delta_2, \theta_2)$.

Similarly, we have

$$
\begin{aligned}
U_2^{\text{ex-post}}(c_1^*) &= \int_0^\infty -\exp\left(-\frac{C_1 t + E_1}{\theta_2}\right)E^P\left[\exp\left(-\frac{D_1 W_t}{\theta_2}\right)\right]dt \\
&= -\exp\left(-\frac{E_1}{\theta_2}\right)\int_0^\infty \exp\left((-\frac{C_1}{\theta_2} + \frac{1}{2}\frac{D_1^2}{\theta_2^2})t\right)dt
\end{aligned}
$$

which converges under Condition (C3) and leads to

$$
U_2^{\text{ex-post}}(c_1^*) = -\frac{\exp\left(-\frac{E_1}{\theta_2}\right)}{\frac{C_1}{\theta_2} - \frac{1}{2}\frac{D_1^2}{\theta_2^2}}.
$$

Proof of Claim 7. Under Condition (C4) , $\widetilde{c}_2^*$ is obtained through the formulas above for $\delta_2 = 0$. We have then

$$
\widetilde{c}_{2,t}^* = C_0 t + D_0 W_t + E_0
$$

with $C_0 = \frac{\theta_2}{\theta_1+\theta_2}\mu$, $D_0 = \frac{\theta_2}{\theta_1+\theta_2}\sigma$ and $E_0 = \frac{1}{2}e_0^* + \frac{1}{2}\frac{\left(\mu\theta_1+\mu\theta_2-\sigma^2\right)(\theta_1-\theta_2)}{\left(\mu\theta_1+\mu\theta_2-\frac{1}{2}\sigma^2\right)}$.

The corresponding ex-ante and ex-post welfare levels for Agent 2 are then given, as above, by

$$\begin{aligned} U_2(\widetilde{c}_2^*) &= -\frac{\exp\left(-\frac{E_0}{\theta_2}\right)}{\frac{C_0}{\theta_2} - \frac{1}{2}\frac{D_0^2}{\theta_2^2} + \delta_2 \frac{D_0}{\theta_2}}, \\ U_2^{\text{ex-post}}(\widetilde{c}_2^*) &= -\frac{\exp\left(-\frac{E_0}{\theta_2}\right)}{\frac{C_0}{\theta_2} - \frac{1}{2}\frac{D_0^2}{\theta_2^2}}, \end{aligned}$$

where $U_2^{\text{ex-post}}(\widetilde{c}_2^*)$ is well defined by construction and where $U_2(\widetilde{c}_2^*)$ is well defined under Condition (C5).