Equilibrium pricing and market completion

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Abstract

In both arbitrage and utility pricing approaches, the *fictitious completion* appears as a very powerful tool that permits to generalize to an incomplete markets framework, results initially established in a complete markets setting. does this technique permit to characterize the equilibrium pricing interval? In other words, does the set of prices that can be reached at the equilibrium for at least one distribution of preferences/endowments and for at least one completion coincide with the set of prices that can be reached at the equilibrium for at least one distribution of preferences/endowments? This note provides a negative answer.

1. Introduction

Asset pricing theory and its link to arbitrage have been formalized by Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), and Duffie and Huang (1986). The main result in these models is that the price process of traded securities is arbitrage free if and only if there exists some equivalent probability measure that transforms it into a martingale, when normalized by the numeraire. When markets are dynamically complete, there is only one such a martingale-probability measure and any contingent claim is priced by taking the expected value of its (normalized) payoff with respect to this measure. It is said priced by arbitrage. When markets are incomplete, arbitrage bounds can be computed taking the expected value of the (normalized) payoff with respect to all the measures that characterize the absence of arbitrage. The obtained arbitrage bounds correspond to the minimum amount it costs to hedge the claim and the maximum amount that can be borrowed against it using dynamic strategies. Each one of the measures that characterize the absence of arbitrage can be seen as associated to a specific *fictitious completion*, that is to say a complete financial market that consists of the initial market completed by additional nonredundant assets. Each price within the arbitrage bounds corresponds then to the arbitrage price in a specific *fictitious* *completion* and the set of such prices is called arbitrage pricing interval. This line of research has been initiated by El Karoui and Quenez (1995).

Utility pricing is another important approach. The price of a given contingent claim, for these theories, corresponds to the amount for which an agent whose preferences and initial endowment are specified is indifferent, at his optimal consumption plan, between a positive or a negative additional marginal quantity of the claim for at that unitary amount. From a mathematical point of view, the problem is to write the agent's utility maximization program and to characterize the marginal utility for consumption at the final date. When markets are complete, Pliska (1986), Cox & Huang (1989, 1991) and Karatzas, Lehoczky & Shreve (1990) adapted the martingale ideas presented above to problems of utility maximization.When markets are incomple, Karatzas et al. (1991) build upon these last references in order to compute, for every *fictitious completion*, the portfolio that maximizes the expected utility of final wealth, and show that that the optimal solution in the initial incomplete market corresponds to the completion which makes the maximum expected utility as small as possible. For a given specification of agent's preferences (utility function and initial endowment) and for a given contingent claim, this approach leads to a unique price. However, this price strongly depends on the chosen specification. In order to obtain preferences and endowments-free bounds, it is possible, for a given contingent claim, to introduce the concept of utility pricing interval (whose bounds are called utility bounds). This interval is the set of utility prices that are obtained for some utility function in the class of von Neumann-Morgenstern (vNM) increasing and concave utility functions and for some initial endowment. Jouini and Kallal (1999) have shown that the utility bounds (utility pricing interval) coincide with the arbitrage bounds (arbitrage pricing interval). In the next we will call them arbitrage/utility bounds (arbitrage/utility pricing interval).

In both arbitrage and utility pricing approaches, the *fictitious completion* appears as a very powerful tool that permits to generalize to an incomplete markets framework, results initially established in a complete markets setting. However, in a classical stochastic volatility setting, Cvitanic et al. (1997) have shown that the arbitrage/utility pricing interval is too large to be of great interest.

It is then necessary to introduce additional conditions in order to derive smaller pricing interval. The equilibrium pricing approach has been introduced first by Bizid et al. (1998) and makes an explicit use of the market clearing conditions. In a general setting and for a given contingent claim, equilibrium bounds delimit the set of all possible equilibrium prices for the asset under consideration for all possible distributions of agents preferences (within the class of vNM increasing and concave utility functions) and for all possible ditributions of initial endowment across the agents. In a diffusion setting, when markets are complete but information (about prices, endowments and preferences) is incomplete or when markets are incomplete but under the assumption that there exists a possibility to complete the market by zero net supply assets without modifying the prices of the already existing ones, Jouini and Napp (2002) show that there exists a unique admissible price for any new asset and this price (as well as the pricing kernel) only depends on the initial assets prices that is to say it does not depend on agents utility functions or initial endowments nor on the choice of the completion. Bizid and Jouini (2005) show that the equilibrium pricing interval is strictly smaller than the arbitrage/utility pricing one. However, they only exploit partial conditions derived from the market clearing ones and the derived pricing interval is then larger than or equal to the equilibrium pricing interval as defined above.

At this stage, a question naturally arises: does the completion technique permit to characterize the equilibrium pricing interval as the set of equilibrium prices associated to all possible completions (that we will call *fictitious completion* equilibrium pricing interval)? In other words, does the set of prices that can be reached at the equilibrium for at least one distribution of preferences/endowments and for at least one completion coincide with the set of prices that can be reached at the equilibrium for at least one distribution of preferences/endowments? Bizid and Jouini (2005) pricing interval appears as strictly larger than the *fictitious completion* equilibrium pricing interval. However, this does not permit to answer to the question since, as underlined above, their interval is larger than the equilibrium pricing interval. The questions above are quite important since the *fictitious completion* equilibrium pricing interval corresponds to the pricing interval considered in Bizid et al. (1998), Jouini and Napp (2002) and Jouini (2003) and has, as shown in these references, many interesting properties in terms of size of robustness.

The present note properly defines these different concepts. In particular, it clarifies the link between the different concepts of equilibrium pricing that can be found in the litterature. We also answer negatively to the questions above and provide a simple example where the set of equilibrium prices associated to all possible vNM utility functions is strictly larger than the set of equilibrium prices associated to all possible *fictitious completion*s.

The paper is organized as follows. Section 2 presents the framework and the main results. Section 3 presents the specific example. Section 4 concludes.

2. The model

We consider a model with 2 dates and finitely many states, where all random variables share a common probability space (Ω, P) in which P is the probability under which all agents evaluate their expected utility and $E[.]$ denotes the associated expectation operator. In the next, we will say that two random variables x and y on (Ω, P) are anticomonotonic (or move in opposite directions) if

$$(x(\omega) - x(\omega'))(y(\omega) - y(\omega')) \leq 0, \ \ P \otimes P \text{ almost surely.}$$

We suppose that there is one consumption good available at date $t = 1$. We assume that we have K firms (K being possibly equal to 0) and firm k produces a random quantity d^k of consumption good at date 1, $k = 1, ...K$. This production is distributed as dividends to shareholders who own the firm. A share θ of firm k insures to the owner a quantity θd^k of the good at date 1. We denote by p^k the price at $t = 0$ of equity claim k, in terms of date-1 consumption goods. We also denote by p the vector $p = (p^k)_{k=1,...,K}$, by d the random vector $d = (d^k)_{k=1,...,K}$ and by $D = \sum_{k=1}^{K} d^k$ the random total supply.

In addition to these equity claims, there are M purely financial assets (that is to say, assets with zero net supply). For $m = 1, ..., M$, the m^{th} financial asset delivers a quantity f^m of consumption good at date $t = 1$ and its date-0 price (in terms of date 1 consumption good) is denoted by q^m. We denote by q the vector $q = (q^m)_{m=1,...,M}$ and by f the random vector $f = (f^m)_{m=1,...,M}$.

In the next we will denote by $\mathcal{M} = ((d,p),(q,f))$ the financial market characterized by the payoffs and prices of the K equity claims and M purely financial assets.

There are N consumers that consume only at date $T = 1$. The n^{th} consumer has a Von Neumann-Morgenstern utility function $U^n(.)$, which associates with any consumption plan C, the following utility level at date 0 :

$$U^n(C) = E[u^n(C)]$$

where u^n maps $\mathbb{R}^{+*}$ in $\mathbb{R}$. We assume the following classical properties on u^n :

Assumption U. For all n, u^n is continuously differentiable, increasing and strictly concave. Moreover, we impose the following Inada condition $u^n(x) \underset{x \to 0^+}{\rightarrow} -\infty$.

The sequence of events is: first, agent n, $n = 1,...,N$, receives an initial endowment in terms of date 1 consumption good as well as in terms of assets denoted by $(w^n, \theta_0^n, \alpha_0^n)$ where w^n is a random variable, $\theta_0^n = \left(\theta_0^{k,n}\right)_{k=1,...,K}$ is a

$K-$dimensional vector and $\alpha_0^n = (\alpha_0^{m,n})_{m=1,\ldots,M}$ is an $M-$dimensional vector and we have $\sum_{n=1}^N \theta_0^n = \mathbf{1}_K$, $\sum_{n=1}^N \alpha_0^n = 0$ and $\sum_{n=1}^N w^n = W$ where $\mathbf{1}_K$ denotes the $K-$dimensional vector whose coordinates are all equal to 1 and where W denotes the random total initial endowment; second, new portfolios and equilibrium prices take place; third states of nature are revealed $(t = 1)$ and agents consume accordingly. The prices come from the equilibrium conditions, and as usual, are considered as given for the agents in their utility maximization program. Before to formally define our equilibrium concept, we need the following definitions.

A consumption-trading strategy S is a vector (C, θ, α) where C is date-1 consumption and where θ (resp. α) is the $K-$dimensional vector of equity claims quantities (resp. $M-$dimensional vector of purely financial assets quantities) owned after trading by the agent who follows strategy S.

We denote by $\alpha \cdot q$ (resp. $\theta \cdot p$) the inner product between α and q (resp. θ and p). The budget constraint of the n^{th} agent is then given by:

$$\begin{aligned} \theta^n \cdot p + \alpha^n \cdot q &= \theta_0^n \cdot p + \alpha_0^n \cdot q \\ C &= w^n + \theta^n \cdot d + \alpha^n \cdot f. \end{aligned} \tag{2.1}$$

For a given agent n with an initial endowment $(w^n, \theta_0^n, \alpha_0^n)$ we can define the convex set of the admissible consumption-trading strategies $\mathcal{A}^n$ as the set of strategies S satisfying the budget constraint 2.1 and the consumption constraint $C \geq 0$.

As usual, an equilibrium and a state-price deflator are defined as follows:

Definition 1. An equilibrium in the economy $\mathcal{E} = \left((d, f), (u^n, w^n, \theta_0^n, \alpha_0^n)_{n=1,...,N} \right)$ is defined by a set of consumption-trading strategies $(C^{n,*}, \theta^{n,*}, \alpha^{n,*})$ and by asset prices (p, q) such that, for $n = 1, ..., N$, $(C^n, \theta^n, \alpha^n)$ is admissible and maximizes U^n on $\mathcal{A}^n$, $\sum_{n=1}^N \theta^n = \mathbf{1}_K$, $\sum_{n=1}^N \alpha^n = 0$ and $\sum_{n=1}^N C^n = W + D$.

Definition 2. A state-price deflator for the economy $\mathcal{E}$ is a random variable ς so that, we have $E[\varsigma] = 1$ and

$$p = E[\varsigma d] \text{ and } q = E[\varsigma f].$$

It is well known that, at the equilibrium, and from the first-order optimal conditions, the random variable

$$\varsigma^n = \frac{(u^n)'(C^{n,*})}{E\left[(u^n)'(C^{n,*})\right]} \tag{3.3}$$

is a state-price deflator for $n = 1, ..., N$.

Furthermore, the price P of any redundant asset paying D at date 1 in terms of consumption good is given by

$$P = E\left[\varsigma^n D\right] \text{ for } n = 1, ..., N.$$

In the next, for a given purely financial asset, we want to characterize the set of possible equilibrium prices that are compatible with the observed assets prices, with the total endowment W as well as with some set of consumers preferences and some distribution of the initial endowment across the agents. This set corresponds to the smaller set of prices that can be inferred from the observation of the initial assets prices (p, q) and from the total endowment W, without further information on the distribution of initial endowment among the agents nor on their preferences (except that they derive from increasing concave vNM utility functions).

Definition 3. In the market $\mathcal{M}$, the equilibrium pricing interval for a zero net supply contingent claim defined by its date-1 payoff f^0 is the set of prices q^0 for which there exists an integer N, a set of increasing and concave utility functions $(u^n)_{n=1,...,N}$, a set of initial endowments $(w^n, \theta_0^n, \alpha_0^n)_{n=1,...,N}$ and a set of consumption trading strategies $(C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*}))$ such that $((C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*}))\,; (p, (q^0, q)))$ is an equilibrium of the economy $\widetilde{\mathcal{E}} =$

$\left(d, \left(f^0, f\right), \left(u^n, w^n, \theta_0^n, \alpha_0^n\right)_{n=1,\ldots,N}\right)$.

The following proposition provides a simpler characterization of the equilibrium pricing interval.

Proposition 1. In the market $\mathcal{M}$ and for a zero net supply contingent claim defined by its date-1 payoff f^0, the equilibrium pricing interval is equal to the set of prices q^0 for which there exists an integer N, a set of consumption plans $\left(C^n\right)_{n=1,\ldots,N}$ and a set of random variables $\left(\varsigma^n\right)_{n=1,\ldots,N}$ such that $\sum_{n=1}^N C^n = W + D$,

$$E\left[\varsigma^n\right] = 1,\ p = E\left[\varsigma^n d\right],\ q = E\left[\varsigma^n f\right],\ q^0 = E\left[\varsigma^n f^0\right] \text{ for } n = 1, ..., N,$$

and such that C^n and ς^n are anticomonotonic for $n = 1, ..., N$.

Proof. If q^0 is in the equilibrium pricing interval for a zero net supply contingent claim defined by its date-1 payoff f^0 then there exists an integer N, a set of increasing and concave utility functions $(u^n)_{n=1,\ldots,N}$, a set of initial endowments $(w^n, \theta_0^n, \alpha_0^n)_{n=1,\ldots,N}$ and a set of consumption trading strategies $(C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*}))$ such that $((C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*}))\,; (p, (q^0, q)))$ is an equilibrium of the economy $\widetilde{\mathcal{E}} = \left(d, \left(f^0, f\right), \left(u^n, w^n, \theta_0^n, \alpha_0^n\right)_{n=1,\ldots,N}\right)$. It suffices to take

$\varsigma^n = \frac{(u^n)'(C^{n,*})}{E\left[(u^n)'(C^{n,*})\right]}$ to derive the result. Conversely, if there exists a set of consumption plans $(C^n)_{n=1,...,N}$ and a set of random variables $(\varsigma^n)_{n=1,...,N}$ that are anticomonotonic for $n = 1, ..., N$ and such that $\sum_{n=1}^{N} C^n = W + D$ and

$$E[\varsigma^n] = 1,\ p = E[\varsigma^n d],\ q = E[\varsigma^n f],\ q^0 = E\left[\varsigma^n f^0\right] \text{ for } n = 1, ..., N,$$

then, by the anticomonotonicity property, it is possible to construct increasing and concave utility functions such that $(u^n)'(C^n) = \varsigma^n$. Let us take $w^n = C^{n,*} - \frac{1}{N}D$, $\theta^{0,n} = \frac{1}{N}\mathbf{1}_K$ and $\alpha^{0,n} = 0$ for $n = 1, ..., N$. It is easy to check that $\left(C^n, \theta^{0,n}, (\alpha^{0,n}, \alpha^n)\right)$ is an optimal consumption-trading strategy for agent n, $n = 1, ..., N$, in the market $\widetilde{\mathcal{M}} = ((d, p), ((q^0, q), f^0, f))$. Since we also have, by construction, $\sum_{n=1}^{n} C^{n,*} = W + D$, $\sum_{n=1}^{n} \theta^{0,n} = \mathbf{1}_K$ and $\sum_{n=1}^{n} \alpha^{0,n} = 0$ then $((C^{n,*}, \theta^{n,*}, (\alpha^{0,n,*}, \alpha^{n,*})); (p, (q^0, q)))$ is an equilibrium of $\widetilde{\mathcal{E}} = (d, (f^0, f), (u^n, w^n, \theta_0^n, \alpha_0^n)_{n=1,...,N})$.

■

When $\mathcal{M}$ is complete, it is well known that there is only one state price deflator denoted by ς^* and we have then that $\varsigma^* = \frac{(u^n)'(C^{n,*})}{E\left[(u^n)'(C^{n,*})\right]}$ for all n and ς^* is then decreasing with $C^{n,*}$ for all n. Consequently ς^* is also decreasing with $\sum_{n=1}^{N} C^{n,*} = W + D$.

We have then the following results :

Corollary 1. When $\mathcal{M}$ is complete, the equilibrium pricing interval of a purely financial asset that pays f^0 at date 0 is given by $[\inf E[\varsigma D], \sup E[\varsigma D]]$ where the infimum and the supremum are taken over the set of all random variables ς such that $E[\varsigma] = 1$, $p = E[\varsigma d]$, $q = E[\varsigma f]$ and such that ς and $W + D$ are anticomonotonic.

Proof. From Proposition 1 and the discussion above, we clearly have that the equilibrium interval is included in the set $[\inf E[\varsigma D], \sup E[\varsigma D]]$ described above. Conversely, let ς such that $E[\varsigma] = 1$, $p = E[\varsigma d]$, $q = E[\varsigma f]$ and $q^0 = E[\varsigma f^0]$ and such that ς and $W + D$ are anticomonotonic and let us take $C^n = \frac{1}{N}(W + D)$ and $\varsigma^n = \varsigma$ for $n = 1, ..., N$. Proposition 1 permits to conclude. ■

In such a complete market setting, Bizid et al. (1998) and Jouini and Napp (2002) have shown that this characterization permits to greatly reduce the size of the pricing interval even if we do not observe all assets prices (incomplete information) while the arbitrage/utility pricing approach does not distinguish between incomplete markets and incomplete information and leads then to larger intervals in the complete markets/incomplete information framework. These kind of properties have also been exploited by Perrakis and Ryan (1984), Ritchken (1985) and Perrakis (1986).

In an incomplete markets setting, Bizid and Jouini (2005) have shown that the characterization of the equilibrium pricing interval provided by Proposition 1 is strictly smaller than the arbitrage/utility pricing interval.

Let us now introduce the concept of *fictitious completion*. Let us assume that there exists J purely financial assets paying $(f^{M+j})_{j=1,\ldots,J}$ with initial prices $(q^{M+j})_{j=1,\ldots,J}$ such that the market $\widetilde{\mathcal{M}}$ that consists in the K equity claims and the $M + J$ purely financial assets is complete. We will say that these newly introduced assets constitute a *fictitious completion* of our initial market. For a purely financial asset whose payoff is given by f^0 we may then consider the equilibrium pricing interval associated to this completion. We may also consider the equilibrium pricing interval associated to all possible completions. Such an interval is formally defined as follows.

Definition 4. In the market $\mathcal{M}$, the *fictitious completion* equilibrium pricing interval for a purely financial asset that pays f^0 at date 1 is the set of all prices q^0 such that there exists J and there exists J purely financial assets paying $(f^{M+j})_{j=1,\ldots,J}$ with initial prices $(q^{M+j})_{j=1,\ldots,J}$ such that the market $\widetilde{\mathcal{M}} = ((p, d), \big((q, (q^{M+j})_{j=1,\ldots,J}), \big(f, (f^{M+j})_{j=1,\ldots,J}\big)\big))$ is complete and q^0 is in the equilibrium pricing interval of the purely financial asset that pays f^0 in the market $\widetilde{\mathcal{M}}$.

The following corollary provides a simple characterization of the *fictitious completion* equilibrium pricing interval.

Corollary 2. In the market $\mathcal{M}$, the *fictitious completion* equilibrium pricing interval for a zero net supply contingent claim defined by its date-1 payoff f^0 is given by $[\inf E[\varsigma D], \sup E[\varsigma D]]$ where the infimum and the supremum are taken over the set of all random variables ς such that $E[\varsigma] = 1$, $p = E[\varsigma d]$, $q = E[\varsigma f]$, and such that ς and $W + D$ are anticomonotonic.

Proof. If q^0 is in the *fictitious completion* equilibrium pricing interval then, by Corollary 1, we have $q^0 = E[\varsigma f^0]$ for ς such that $E[\varsigma] = 1$, $p = E[\varsigma d]$, $q = E[\varsigma f]$ and such that ς and $W + D$ are anticomonotonic. Conversely, if we have such a q^0, let us add to our market enough purely financial assets defined by their payoffs $(f^{M+j})_{j=1,...,J}$ and by their initial prices $q^{M+j} \equiv E\left[\varsigma f^{M+j}\right]$, $j = 1, ..., J$ and let us denote by $\widetilde{\mathcal{M}}$ this completion. It is easy to check that q^0 is in the equilibrium pricing interval of the purely financial asset that pays f^0 in the market $\widetilde{\mathcal{M}}$. ■

In a diffusion setting, Jouini and Napp (2002) have shown that the *fictitious completion* equilibrium pricing interval is reduced to a unique price that corresponds to Föllmer and Schweitzer (1991) price when equity prices are increasing functions of dividends[1]. Recall that this price corresponds to the price obtained through the Arrow-Debreu price system that does not price "orthogonal" (with respect to the market spanned by the initial assets) risks.

The equilibrium approach seems then very promising in order to obtain tight pricing intervals.

3. A Specific example

We know from Bizid and Jouini (2005) that the equilibrium pricing interval is, in general, strictly smaller than the arbitrage/utility pricing interval. Furthermore, we know that the arbitrage/utility pricing interval coincides with the *fictitious completion* arbitrage/utility pricing interval defined as the set of all arbitrage/utility prices that are obtained in at least a completion.

Does the *fictitious completion* equilibrium pricing interval coincide with the equilibrium pricing interval.

The next example answers negatively to this question.

Let us consider a model with 2 agents ($N = 2$), 4 equiprobable states of the world and such that $K = M = 0$ and a total endowment $W = (2.997, 2.52, 2.501, 2)$.

Our aim is to price a purely financial asset whose payoff is given by $f^0 =$

[1] Jouini and Napp (2003) provides theoretical justifications for such an assumption.

$(1,0,0,\frac{1}{10})$.

Let us take $w^1=(1.499,1.5,1.001,1)$, $w^2=(1.498,1.02,1.5,1)$, $\varsigma^1=(1.137\,5,0.502\,56,1.14,1.22)$ and $\varsigma^2=(1.137\,3,1.1415,0.5,1.221\,2)$. Since w^n and ς^n are anticomonotonic for $n=1,2$, it is possible to construct u^1 and u^2 such that $(u^1)'(w_i^1)=\varsigma_i^1$ and $(u^2)'(w_i^2)=\varsigma_i^2$ for $i=1,...,4$.

Remark that $E[\varsigma^1]=E[\varsigma^2]=1$ and $E[\varsigma^1 f^0]=E[\varsigma^2 f^0]=0.31485$ which means that $q^0=0.31485$ is an equilibrium price for the asset under consideration : at that price each agent has a zero demand for this asset and his optimal demand is equal to his initial endowment which ensures that markets clear.

Now let us consider the price P of our asset in a given completion by purely financial (zero net supply) assets. Then, by Corollary 2, we have $P\in[\inf E[\varsigma D],\sup E[\varsigma D]]$ for ς such that $E[\varsigma]=1$ and such that ς is decreasing with $W=w^1+w^2=(2.997,2.52,2.501,2)$ that is to say such that $\varsigma_1\leq\varsigma_2\leq\varsigma_3\leq\varsigma_4$. For such a ς, it is easy to check that we necessarily have $\frac{1}{30}=0\times1+\frac{1}{3}\times0+\frac{1}{3}\times0+\frac{1}{3}\times\frac{1}{10}\leq E[\varsigma D]\leq\frac{1}{4}\times(1+0+0+\frac{1}{10})=0.275$ and that any value within these bounds can be reached by such a ς.

There is then prices in the equilibrium pricing interval $EPI(f^0)$ (namely, $q^0=0.31485$) that are not in the *fictitious completion* equilibrium pricing interval $FCEPI(f^0)$.

In other words, the set of equilibrium prices associated to some completion and to some distribution of the total wealth W among the agents is strictly smaller than the set of equilibrium prices associated to some distribution of the total wealth W.

Note that the arbitrage/utility pricing interval $AUPI\left(f^0\right)$ in our framework is given by $\left[\inf E\left[\zeta f^0\right], \sup E\left[\zeta f^0\right]\right]$ for ζ such that $E\left[\varsigma\right] = 1$ and we have then $AUPI\left(f^0\right) = \left[0, 1\right].$

Furthermore, any price in the equilibrium pricing interval $EPI(f^0)$ corresponds to the equilibrum price in a given economy. Since we have $W_1 > W_2$, at least one agent in this economy satisfies $C_1^{n,*} > C_2^{n,*}$ which gives $\zeta_1^n < \zeta_2^n$ for the state price deflator associated to that agent and the price of f^0 is then necessarily smaller than 0.5.

We have then, in our example, $\left[\frac{1}{30}, \frac{11}{40}\right] = FCEPI(f^0) \subsetneq EPI(f^0) \subset [0, 5] \subsetneq AUPI\left(f^0\right) = \left[0, 1\right].$

4. Conclusion

Bizid and Jouini have shown that their pricing interval obtained through partial equilibrium conditions is strictly smaller than the arbitrage/utility pricing interval and strictly larger than the *fictitious completion* equilibrium pricing interval.

However, it was unclear if the full exploitation of the equilibrium conditions would permit to reach a smaller pricing interval and wether this interval would be equal to the *fictitious completion* equilibrium pricing interval. This question worths attention since it has been shown by Bizid et al. (1998) and Jouini and Napp (2002) that the *fictitious completion* equilibrium pricing interval might be quite small and it has been shown by Jouini (2003) that the *fictitious completion* equilibrium pricing interval has good convergence properties and is robust to small perturbations on the characteristics of the economy. In this note, we show that even when the equilibrium conditions are fully exploited, the resulting equilibrium pricing interval might remain strictly larger than the *fictitious completion* equilibrium pricing interval. Then, work remains to be done in order to analyze to which extent Bizid and Jouini (2005) bounds can be reduced further.

The main conclusion of this note is that the *fictitious completion* technique does not permit to scan all possible equilibrium prices while it permits to scan all arbitrage/utility prices.

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